

# Topological Completeness of the Transfinite Provability Logic\*

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## Abstract

Let  $\Lambda$  be an ordinal. The polymodal provability logic  $\text{GLP}_\Lambda$  contains modalities  $\langle \lambda \rangle$  for  $\lambda < \Lambda$  intended to capture progressively stronger notions of consistency in mathematical theories. We show  $\text{GLP}_\Lambda$  is complete with respect to its topological interpretation, where each modality  $\langle \xi \rangle$  denotes the derived-set operator in some topology  $\mathcal{T}_\xi$ .

Specifically, for each ordinal  $\Lambda$  and any tall-enough scattered space  $(X, \tau)$ , one defines topologies  $\{\mathcal{T}_i\}_{i < \Lambda}$  that are to  $\tau$  as 'iterated order topologies' are to  $\text{Ord}$ . We show that, if we restrict the domain of valuations or further refine  $\{\mathcal{T}_i\}_{i < \Lambda}$ , then the logic of the resulting polytopological space is  $\text{GLP}_\Lambda$ .

## 1 Introduction

We study topological models of the polymodal provability logic  $\text{GLP}_\Lambda$  (where  $\Lambda$  is to be understood as a parameter; by 'GLP,' we refer to any unspecified logic of this form). The logic, the natural transfinite extension of Japaridze's polymodal logic  $\text{GLP}_\omega$ , is used to simultaneously model various degrees of provability in mathematical theories (see, e.g., [6, 14, 18]). As the logic is incomplete for any class of relational semantics, the most natural place in which one might expect to find models is neighborhood semantics, which are in fact polytopological spaces—structures consisting of sets equipped with several topologies. It was conjectured by L. Beklemishev and D. Gabelaia [10] that GLP is complete with respect to a natural class of spaces called the *canonical*  $\text{GLP}_\Lambda$ -spaces, although it follows from the work of J. Bagaria [4], that large-cardinal assumptions would be needed.

Nevertheless it is known that there are often other natural polytopologies that can be used in lieu of the canonical GLP-spaces as models for GLP, even in ZFC. Indeed, this has been done for countable  $\Lambda$  by D. Fernández-Duque [15], building on the work of L. Beklemishev and D. Gabelaia [9]. Our work extends those results in two directions: first and foremost, we prove completeness of  $\text{GLP}_\Lambda$  without restricting  $\Lambda$  in any way. Additionally, this completeness is obtained with respect to individual spaces that need not be ordinal numbers.

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The remaining challenge in extending previous results into the uncountable consists in identifying suitable combinatorial properties of ordinal numbers, which will be easily stated by employing an ordinal notation system developed by D. Fernández-Duque and J. Joosten [17] based on the so-called *hyperexponential* functions. Here, 'combinatorial' is not to be understood in the usual sense; instead, this 'combinatorics' has a more local flavor. In fact, the heart of our proof (see Lemma 4.21) consists of a deep investigation into the arithmetical structure of the ordinals and the topological properties it induces.

The spaces that provide our semantics are reviewed in Section 3. Our main theorems are precisely stated and proved in Section 4. They state that the topology of any sufficiently tall space can be extended to yield a polytopology with respect to which the logic  $\text{GLP}_\Lambda$  is topologically complete. Moreover, these extensions can be nicely controlled if we restrict the domain of topological valuations. An introduction to the problem and a review of the terminology involved are given in the next section. In Section 5, we prove a result in the same vein as the main completeness theorems. This result (Theorem 5.3) states that any formula consistent with GLP can be satisfied in a 'finite-support' analog of the spaces constructed in the preceding sections. This result is stated in such a way that, when the underlying space is an ordinal number, the resulting polytopology is built over the order topology. We close in Section 6 with some remarks and open questions.

## 2 The Polymodal Logic of Provability

For any ordinal number  $\Lambda$  we consider a language  $\mathcal{L}_\Lambda$  consisting of a countable set of propositional variables  $\mathbb{P}$  together with the constants  $\top, \perp$ ; boolean connectives  $\wedge, \vee, \neg, \rightarrow$ ; and a modality  $[\xi]$  for each ordinal  $\xi < \Lambda$ . As usual, we write  $\langle \xi \rangle$  as a shorthand for  $\neg[\xi]\neg$ .

**Definition 2.1.** The logic  $\text{GLP}_\Lambda$  is then defined to be the least logic containing all propositional tautologies and the following axiom schemata:

- (i)  $[\xi](\varphi \rightarrow \psi) \rightarrow ([\xi]\varphi \rightarrow [\xi]\psi)$  for all  $\xi < \Lambda$ ,
- (ii)  $[\xi]([\xi]\varphi \rightarrow \varphi) \rightarrow [\xi]\varphi$  for all  $\xi < \Lambda$ ,
- (iii)  $[\xi]\varphi \rightarrow [\zeta]\varphi$  for all  $\xi < \zeta < \Lambda$ ,
- (iv)  $\langle \xi \rangle \varphi \rightarrow [\zeta]\langle \xi \rangle \varphi$  for all  $\xi < \zeta < \Lambda$ ,

and closed under the rules modus ponens and necessitation for each  $[\xi]$ :

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \text{ MP} \qquad \frac{\varphi}{[\xi]\varphi} \text{ nec}$$

Note that GLP when restricted to any one modality is simply the well-known logic GL. Modal logics are usually studied by means of relational semantics. A *Kripke  $\Lambda$ -frame* is a structure  $(W, \{R_\xi\}_{\xi < \Lambda})$ , where each  $R_\xi$  is a binary relation on  $W$ . We define a *valuation*  $\llbracket \cdot \rrbracket$  to be a function assigning subsets of  $W$  to each  $\mathcal{L}_\Lambda$ -formula such that  $\llbracket \cdot \rrbracket$  respects boolean connectives and such that

$$\llbracket \langle \xi \rangle \varphi \rrbracket = R_\xi^{-1} \llbracket \varphi \rrbracket.$$

**Proposition 2.2** ([20]).  $\text{GLP}_1$  is complete with respect to the class of finite relational structures  $(W, R)$  that are conversely wellfounded trees.

The preceding proposition provides a convenient way to study  $\text{GLP}_1$ . However, as is well known,  $\text{GLP}_\Lambda$  is incomplete with respect to any class of relational structures whenever  $1 < \Lambda$ . This motivates the search for topological models of the GLP.

Recall that  $x$  is a *limit point* of  $A$  if  $A$  intersects every punctured neighborhood of  $x$ . We call the set of limit points of  $A$  the *derived set* of  $A$  and denote it by  $dA$ . We may also denote it by  $d_\tau A$  to emphasize the topology we are considering. The derived set operator is iterated transfinitely by setting

1.  $d^0 A = A$ ,
2.  $d^{\alpha+1} A = d^\alpha dA$ , and
3.  $d^\gamma A = \bigcap_{\alpha < \gamma} d^\alpha A$  for limit ordinals.

Since  $d^\alpha X \supset d^\beta X$  whenever  $\alpha < \beta$ , there exists a minimal ordinal  $ht(X)$ —the *height* of  $X$ —such that  $d^{ht(X)} X = d^{ht(X)+1} X$ . For any  $x \in X$ , we let  $\rho_\tau x$ , the *rank* of  $x$ , be the least ordinal  $\xi$  such that  $x \notin d^{\xi+1} X$ , if it exists.

Throughout this paper, we will speak about *rank-preserving* extensions of topologies. The following characterization was noted in [9]:

**Lemma 2.3.** *A topology  $\sigma$  is a rank-preserving extension of a scattered topology  $\tau$  if, and only if,  $\rho_\tau(U)$  is an initial segment of  $\text{Ord}$  for each  $U \in \sigma$ .*

A point in  $A$  that is not a limit point is *isolated*. Thus a point is isolated if and only if it has rank 0. We denote by  $iso(A)$  the set of isolated points in  $A$ . A topological space is *scattered* if  $iso(A) \neq \emptyset$  for each  $A \subset X$  (alternatively, if  $d^{ht(X)} X = \emptyset$ ).

We study *polytopological spaces*—structures  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$ , where  $X$  is a set and  $\{\mathcal{T}_\iota\}_{\iota < \Lambda}$  is a sequence of topologies of length  $\Lambda$ . We also consider the more general *ambiances*:

**Definition 2.4** (Ambiance). An *ambiance* is a structure  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$ , where  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  is a polytopological space and  $\mathcal{A} \subset \wp(X)$  is a Boolean algebra closed under  $d_{\mathcal{T}_\xi}$ , for all  $\xi < \Lambda$ .

This notion was introduced in [15], in whose main completeness proof it figured. For the purposes of modeling (poly)modal logics, ambiances can indeed be regarded as a generalization of polytopological spaces. Therefore, we will often identify a space  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  with the ambiance  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \wp(X))$ . Topological semantics for modal logics may be defined by interpreting diamonds as topological derivatives.

**Definition 2.5** (Topological semantics). Let  $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  be an ambiance. A *valuation* is a function  $\llbracket \cdot \rrbracket : \mathcal{L}_\Lambda \rightarrow \mathcal{A}$  such that for any  $\mathcal{L}_\Lambda$ -formulae  $\varphi, \psi$ :

- (i)  $\llbracket \perp \rrbracket = \emptyset$ ;
- (ii)  $\llbracket \neg \varphi \rrbracket = X \setminus \llbracket \varphi \rrbracket$ ;
- (iii)  $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ ;

$$(iv) \llbracket \langle \xi \rangle \varphi \rrbracket = d_{\tau_\xi} \llbracket \varphi \rrbracket.$$

A *model*  $\mathfrak{M} = (\mathfrak{X}, \llbracket \cdot \rrbracket)$  is an *ambience* together with a valuation. We say that  $\varphi$  is *satisfied* in  $\mathfrak{M}$  if  $\llbracket \varphi \rrbracket$  is nonempty and we say  $\varphi$  is *valid* in an *ambience*  $\mathfrak{X}$  and write  $\mathfrak{X} \models \varphi$  if  $\llbracket \varphi \rrbracket = X$  for any model based on  $\mathfrak{X}$ . We define the *logic* of an *ambience* to be the set

$$\mathcal{L}(\mathfrak{X}) = \{\varphi \in \mathcal{L}_\Lambda : \mathfrak{X} \models \varphi\}.$$

In order that an *ambience* validate the axioms of GLP, we need to impose some regularity conditions (see [8]). An *ambience*  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  is a  $\text{GLP}_\Lambda$ -*ambience* if  $\{\mathcal{T}_\iota\}_{\iota < \Lambda}$  is non-decreasing scattered and

$$d_\xi A \in \mathcal{T}_\zeta \text{ for all } \xi < \zeta \text{ and all } A \in \mathcal{A} \quad (2.1)$$

A sequence  $\{\mathcal{T}_\iota\}_{\iota < \Lambda}$  is a  $\text{GLP}_\Lambda$ -*polytopology* if there exists a Boolean algebra  $\mathcal{A} \subset \wp(X)$  such that  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  is a  $\text{GLP}_\Lambda$ -*ambience*.

**Lemma 2.6.** *Any  $\text{GLP}_\Lambda$ -ambience validates all theorems of  $\text{GLP}_\Lambda$ .*

A natural way of constructing  $\text{GLP}_\Lambda$ -polytopologies appears to be to start with any scattered topology and simply add all derived sets at each stage, thus making  $\mathcal{A} = \wp(X)$ . This results in what has come to be known as the *canonical GLP-space generated by X*, although it might not be a desirable alternative, as the topologies quickly become extremely fine. In fact, for the most natural examples, their non-discreteness becomes undecidable within ZFC after two or three iterations.

One way out of this, explored in [9], is to extend the topology at each stage before adding derived sets. Extending the topology reduces the amount of derived sets attainable and makes subsequent topologies coarser. A different approach, introduced in [15], is to fix increasing topologies from the beginning and choosing an appropriate algebra that ensures condition (2.1) is verified. We will consider both approaches in our completeness proofs. They correspond, respectively, to the use of *limit-maximal* topologies<sup>1</sup> and *ambiences* distinct from the whole powerset. Each of these approaches results in one of our main results: Theorems 4.19 and 4.20 below. Their statements use an ordinal notation reviewed in Section 4—roughly, define  $e\beta$  to be 0 if  $\beta$  is 0, and  $\omega^\beta$  otherwise; we write  $e^\alpha\beta$  to mean that the function  $e$  applied to  $\beta$  is iterated  $\alpha$ -many times. Recall that the *weight* of a topology, denoted  $w(\cdot)$ , is the least cardinality of a basis.

**Theorem 1** (Theorem 4.19). *Let  $\Lambda$  be any ordinal and  $(X, \tau)$  be any scattered space such that  $e^{e^2\Lambda}1 \leq ht(X)$ . Then, there exist a Boolean algebra  $\mathcal{A}$  over  $X$  and refinements  $\mathcal{T}_\lambda \supset \tau$  of weight  $w(\tau) + |\Lambda|$ , for  $\lambda < \Lambda$ , such that*

$$\mathcal{L}(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A}) = \text{GLP}_\Lambda.$$

In fact, the polytopologies given by Theorem 4.19 will have a very simple description. The bound is also not very large—for instance,  $|e^{e^2\Lambda}1| = |\Lambda| + \aleph_0$ . In some sense—namely, with respect to  $\mathcal{T}_0$ —the bound is sharp. The question of finding a sharp bound (with respect to  $\tau$ ) is nontrivial (and perhaps inessential).

<sup>1</sup>See Definition 3.2.

It is not hard to show from known results that it suffices to assume  $e^{1+\Lambda}1 \leq ht(X)$  whenever  $\Lambda$  is a countable limit. This is not so clear for the general case, although the following can be shown for uncountable  $\Lambda$ :

- If  $\Lambda$  is multiplicatively indecomposable (which can be taken to be true without any downside), then it suffices to assume  $e^{\Lambda \cdot 2}1 \leq ht(X)$ .
- For some natural spaces, it suffices to assume  $e^\Lambda 1 \leq ht(X)$ . We construct such spaces in the proof.
- It is not hard to see that those are the only possibilities for the constructions in this paper (i.e., the lower bound can only be either  $e^\Lambda 1$  or  $e^{\Lambda \cdot 2}1$ ) if  $\Lambda$  is additively indecomposable. In Section 5, we reformulate the possibility of it being  $e^\Lambda 1$  in terms of the existence of certain functions on  $\text{Ord}$ .

The spaces we construct are based on the *generalized Icard topologies*  $\mathcal{I}_\lambda$ , which resemble the topologies originally introduced to provide semantics for the variable-free fragment of GLP (see [16, 19]). The following analogous result is presented with the less stringent condition  $\mathcal{A} = \wp(X)$ . This requires, however, surrendering the simple description of the topologies:

**Theorem 2** (Theorem 4.20). *Let  $\Lambda$  be any ordinal and  $(X, \tau)$  be a scattered space such that  $e^{e^2 \Lambda \cdot 2}1 \leq ht(X)$ . Then, there exist refinements  $\mathcal{T}_\lambda \supset \tau$ , for  $\lambda < \Lambda$ , such that*

$$\mathcal{L}(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}) = \text{GLP}_\Lambda.$$

The polytopologies given by Theorem 4.20 are also based on generalized Icard topologies. These topologies are easy to work with and, as mentioned before, reduce the issue of completeness to combinatorial properties of ordinal numbers. The downside is that they in general do not validate the axioms of GLP—a problem that will need to be circumvented.

We will make use of a reduction of  $\text{GLP}_\Lambda$  to  $\text{GLP}_\omega$  in the following way: for each  $\mathcal{L}_\Lambda$ -formula  $\varphi$  where modalities  $[\lambda_0], \dots, [\lambda_{n-1}]$  appear, define its *condensation* to be the  $\mathcal{L}_n$ -formula  $\varphi^c$  that results by uniformly substituting  $[k]$  for  $[\lambda_k]$ . The following reduction lemma holds:

**Lemma 2.7** (see, e.g., [15]). *Let  $\varphi$  be a  $\mathcal{L}_\Lambda$ -formula where  $n$  modalities appear. Then  $\text{GLP}_n \vdash \varphi^c$  implies  $\text{GLP}_\Lambda \vdash \varphi$ .*

The converse to Lemma 2.7 is obtained as a consequence of the proof of Theorem 4.19, although the result was known before.

### 3 GLP-ambiances

We begin by introducing the topologies on which our models will be based:

**Definition 3.1** (Generalized Icard Topologies). Let  $(X, \tau)$  be a scattered space of rank  $\Theta$ . We define a topology  $\tau_{\uparrow 1}$  generated by  $\tau$  and all sets of the form

$$(\alpha, \beta)^\tau := \{x \in X : \alpha < \rho_\tau x < \beta\},$$

for ordinals  $\alpha < \beta \leq \Theta + 1$ . We iterate this construction by setting

- $\tau_{\uparrow(\iota+1)} = (\tau_{\uparrow\iota})_{\uparrow 1}$ , and
- $\tau_{\uparrow\lambda} = \bigcup_{\xi < \lambda} \tau_{\uparrow\xi}$  at limit stages.

These are called the *generalized Icard topologies*.

These topologies were defined differently in [3]. By Lemma 4.12.1 below, both definitions coincide. Another equivalent formulation is as follows:  $\tau_{\uparrow 1}$  is generated by  $\tau$  and the family

$$\{d^\xi X : \xi \in \text{Ord}\}. \quad (3.1)$$

The topologies defined by (3.1) were studied in [9], where they were motivated by Lemma 3.4 below.

**Definition 3.2** (limit-maximal extension). Let  $(X, \tau)$  be a scattered space. We say that  $\tau_*$  is a *limit extension* of  $\tau$  if

1.  $\tau \subset \tau_*$ ,
2.  $\rho_{\tau_*} = \rho_\tau$ , and
3. the identity function  $id : (X, \tau) \rightarrow (X, \tau_*)$  is continuous at all points of successor rank.

We say that  $\tau_*$  is a *limit-maximal topology* if there are no proper limit extensions of  $\tau_*$ .

As can be shown using Zorn's lemma, any scattered space has a limit-maximal extension. The following characterization from [9] is essential.

**Lemma 3.3.**  *$(X, \tau)$  is a limit-maximal space if, and only if, for all  $x \in X$  of limit rank and all open  $V$  consisting of points of rank below  $\rho_\tau(x)$ , one of the following occurs:*

1.  $V \cup \{x\} \in \tau$ , or
2. there exists a  $\tau$ -neighborhood  $U$  of  $x$  such that  $\rho_\tau(U \cap V) < \rho_\tau(x)$ .

Among the applications of Lemma 3.3 is the following result:

**Lemma 3.4.** *Suppose  $(X, \tau)$  is limit-maximal and  $\lambda$  is an ordinal. Then  $\{dA : A \subset X\} \subset \tau_{\uparrow\lambda}$ .*

**Definition 3.5** (Beklemishev-Gabelaia space). Let  $(X, \tau)$  be a scattered space. We say that a polytopological space  $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  is a *BG-space based on  $\tau$*  if

1.  $\mathcal{T}_0$  is a limit-maximal extension of  $\tau$ ,
2.  $\mathcal{T}_{\xi+1}$  is a limit-maximal extension of  $(\mathcal{T}_\xi)_{\uparrow 1}$ , except perhaps if it is the last topology in the sequence,
3.  $\mathcal{T}_\lambda$  is a limit-maximal extension of  $\bigsqcup_{\xi < \lambda} \mathcal{T}_\xi$  for limit  $\lambda$ .

It follows from Lemma 3.4 that if  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  is a BG-space, then the structure  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \wp(X))$  is a  $\text{GLP}_\Lambda$ -ambiance. We are faced with a dichotomy between two natural candidates for constructing natural  $\text{GLP}_\Lambda$ -ambiances. We can either have topologies be as fine as possible at each stage, so that we allow the algebra to be larger, or we can choose topologies as coarse as possible and maintain a nice structure on them, with the drawback that the algebras taken be small.

**Definition 3.6** (Maximal/minimal ambiance). We say a  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  is *maximal* if the underlying polytopology is a BG-space and  $\mathcal{A} = \wp(X)$ . We say that  $\mathfrak{X}$  is *minimal* if the topologies satisfy  $\mathcal{T}_{(\lambda+\mu)} = (\mathcal{T}_\lambda)_{\uparrow\mu}$ .

The completeness theorems will state that any sufficiently tall topological space can be extended to a maximal (resp. minimal) ambiance with respect to which GLP is complete. It is immediate from Definition 3.1 that the minimality condition  $(\tau_{\uparrow\lambda})_{\uparrow\mu} = \tau_{\uparrow(\lambda+\mu)}$  is verified for any scattered space  $(X, \tau)$ .

**Definition 3.7.** We fix some notation related to ordinal arithmetic.

1. Whenever  $\alpha < \beta$ , we denote by  $-\alpha + \beta$  the unique ordinal  $\gamma$  such that  $\alpha + \gamma = \beta$ .
2. Whenever  $A$  is a set of ordinals, we denote by  $\alpha + A$  the set  $\{\alpha + \beta : \beta \in A\}$ . Expressions such as  $-\alpha + A$  are defined analogously, if they make sense.
3. For all nonzero  $\xi$ , there exist ordinals  $\alpha$  and  $\beta$  such that  $\xi = \alpha + \omega^\beta$ . Such a  $\beta$  is unique. We denote it by  $\ell\beta$  and call it the *end-logarithm* of  $\xi$ .
4. For all nonzero  $\xi$ , there exists a unique ordinal  $\eta$  such that  $\xi$  can be written as  $\omega^\eta + \gamma$ , with  $\gamma < \xi$ . We denote this ordinal by  $L\xi$  and call it the *initial logarithm* of  $\xi$ .
5. For any ordinal  $\xi$ , write  $\mathcal{L}\xi$  for the order type of  $\text{Lim} \cap [1, \xi)$ .

The operations  $\ell$ ,  $L$ , and  $\mathcal{L}$  should be regarded as functions on (a sufficiently large subset of)  $\text{Ord}$ . Nonetheless, in its use and in general whenever we deem it convenient, we will omit the symbol ' $\circ$ ' for function composition, as well as perhaps parentheses.

**Lemma 3.8.**  $\mathcal{L}$  is a normal<sup>2</sup> function given recursively by

- $\mathcal{L}(\alpha + \beta) = \mathcal{L}\alpha + \mathcal{L}\beta$ ;
- $\mathcal{L}(\omega^\omega \cdot \alpha) = \omega^\omega \cdot \alpha$ ;
- $\mathcal{L}\omega^{n+1} = \omega^n$ .

Consequently,  $\mathcal{L}$  is the function  $\omega \cdot (1 + \alpha_0) + k \mapsto \alpha_0$ .

*Proof.* Note that  $\mathcal{L}: \text{Lim} \rightarrow \text{Ord}$  is an order-preserving isomorphism. From this follows that  $\mathcal{L}$  is normal. Clearly,  $\mathcal{L}(\alpha + \beta) = \mathcal{L}\alpha + \mathcal{L}\beta$ . A simple induction using this fact and normality establishes  $\mathcal{L}\omega^{n+1} = \omega^n$ , for  $\mathcal{L}\omega = 1$  and

$$\mathcal{L}\omega^{n+1} = \mathcal{L}(\lim_{k \rightarrow \omega} \omega^n \cdot k) = \lim_{k \rightarrow \omega} \mathcal{L}(\omega^n \cdot k) = \lim_{k \rightarrow \omega} (\omega^{n-1} \cdot k) = \omega^n.$$

By this and normality,  $\mathcal{L}(\omega^\omega \cdot \alpha) = \omega^\omega \cdot \alpha$ . □

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<sup>2</sup>I.e., continuous and increasing.

Our completeness proof will heavily rely upon a fine analysis of generalized Icard topologies and their structure induced by the arithmetical properties of ordinals. Hence, developing a thorough intuition about them will be crucial. A most useful remark in this direction is the fact that they are to arbitrary topological spaces as the usual order topology is to ordinal numbers. Indeed, define the *initial segment topology*  $\mathcal{I}_0$  on an ordinal  $\Theta$  (or on  $\text{Ord}$ ) to be generated by all initial segments  $[0, \alpha]$ , for  $\alpha < \Theta$ . Then  $(\Theta, \mathcal{I}_0)$  is a scattered space: a rather trivial scattered space—it carries no further information than the usual ordering on  $\text{Ord}$ . For instance, we have  $\rho_{\mathcal{I}_0}\alpha = \alpha$  for all  $\alpha$  and  $ht(\Theta, \mathcal{I}_0) = \Theta$ .

**Lemma 3.9.**  $\mathcal{I}_1 := \mathcal{I}_{0 \uparrow 1}$  is the order topology. We have  $\rho_{\mathcal{I}_1}\alpha = \ell\alpha$  for all  $\alpha$ , so in particular isolated points are exactly the successor ordinals. Moreover,  $ht(\Theta, \mathcal{I}_1) = L(\Theta) + 1$ .

*Proof.* It is not hard to see that  $\mathcal{I}_1$  is the order topology, and that the rank function is  $\ell$  is established by a simple induction. Finally, let  $\mathbb{H}$  be the class of additively indecomposable ordinals. It follows that

$$ht(\Theta, \mathcal{I}_1) = \sup_{\xi < \Theta} (\ell\xi + 1) = \sup_{\xi < \Theta \cap \mathbb{H}} (\ell\xi + 1) = \sup_{\xi < \Theta \cap \mathbb{H}} (L\xi + 1) = L(\Theta) + 1,$$

since for any  $\omega^\eta < \Theta$ , we can write  $\Theta$  in the form  $\omega^\eta + (-\omega^\eta + \Theta)$ .  $\square$

In what follows, we write simply  $\mathcal{I}_\lambda$  instead of  $\mathcal{I}_{0 \uparrow \lambda}$ . As we will see, the proofs of the main theorems can be quickly reduced to the case when the underlying space is an ordinal equipped with a topology of the form  $\mathcal{I}_\lambda$ , for—once this is done—Corollary 4.17 will allow us to ‘shift’ the resulting ambiance to a BG-space over more general topological spaces.

**Definition 3.10** (ordinal BG-space). A BG-space  $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  over  $(X, \tau)$  is called an *ordinal BG-space* if  $X$  is an interval of the ordinals and  $\tau = \mathcal{I}_\varsigma$  for some  $\varsigma$ .

We are ready to outline the completeness proof for  $\text{GLP}_\Lambda$  for ordinal BG-spaces. We will start with a scattered space of the form  $([1, \Theta], \mathcal{I}_\Sigma)$ , with  $\Sigma \geq \Lambda$  and  $\Theta$  large enough. Assume, without loss of generality,  $\Sigma = \Lambda$ . We will then make the choice whether to construct a minimal ambiance on top of  $([1, \Theta], \mathcal{I}_\Lambda)$ , or a maximal one. If we opt for a minimal ambiance, then we will take  $\{\mathcal{I}_{\Lambda+\iota}\}_{\iota < \Lambda}$  as the polytopology. The construction of the algebra will need specific care. After this has been taken care of, we will have proved the corresponding instance of Theorem 1.

If, on the other hand, we opt for a maximal ambiance, then the algebra will pose no problem, but we will need to carefully engineer the underlying BG-space in order that the mechanism of the embedding lemma still work. Once we do this, we will have proved Theorem 2. This will be detailed in the following section.

## 4 Completeness

We prove the completeness theorems. The key points of the proof are a *pullback construction*, an *Embedding Lemma*, and a *Product Lemma*. In order to state them, we need to introduce additional terminology.



## 4.1 d-maps and J-maps

There is also an appropriate notion of structure-preserving mappings between scattered spaces. We say that a function between topological spaces is *pointwise discrete* if the preimage of any singleton is a discrete subspace.

**Definition 4.1** (d-map). Let  $X$  and  $Y$  be scattered spaces. A function  $f : X \rightarrow Y$  is a *d-map* if it is continuous, open, and pointwise discrete.

Clearly, any homeomorphism is a d-map. In particular, ordinal addition and subtraction, i.e., functions of the form

$$(-\xi + \cdot) : ([\xi, \xi + \Theta], \mathcal{I}_\zeta) \rightarrow ([0, \Theta], \mathcal{I}_\zeta),$$

are d-maps. The rank function

$$\rho_\tau : (X, \tau) \rightarrow ([0, \rho_\tau X], \mathcal{I}_0)$$

is also a d-map. A more interesting example is given by end-logarithms of the form:

$$\ell : (\Theta, \mathcal{I}_{\zeta+1}) \rightarrow (\Theta, \mathcal{I}_\zeta).$$

A proof of this, and the more general Lemma 4.13 below can be found in [15].

Since the composition of d-maps is a d-map, they can be thought of as morphisms in the category of scattered spaces. We will now state various properties of d-maps.

**Lemma 4.2.** *Let  $f : X \rightarrow Y$  be a d-map.*

1. *If  $Y$  is an ordinal  $\Theta$  with the initial segment topology, then  $f$  is the rank function on  $X$ .*
2. *For any  $A \subset Y$ ,  $f^{-1}dA = df^{-1}A$ .*
3.  *$f : (X, \tau_{\uparrow\lambda}) \rightarrow (Y, \sigma_{\uparrow\lambda})$  is a d-map for any  $\lambda$ .*
4. *If  $f$  is surjective, then for any  $\mathcal{L}_\Lambda$ -formula  $\varphi$ ,  $X \models \varphi$  implies  $Y \models \varphi$ .*

*Proof.* Items 1 and 2 appear in [9]; item 4 appears in [11] in the current formulation. Item 3 is proved in [3], but therein a different definition of  $\tau_{\uparrow\lambda}$  is used, and we still have not shown that they are equivalent. Nonetheless, the claim can be proved by an easy induction.

Suppose  $f : (X, \tau_{\uparrow\lambda}) \rightarrow (Y, \sigma_{\uparrow\lambda})$  is a d-map. Note that 2 implies that d-maps are rank-preserving, i.e.,

$$\rho_{\tau_{\uparrow\lambda}}(x) = \rho_{\sigma_{\uparrow\lambda}}(f(x)) \text{ for every } x \in X.$$

It follows that for any  $\tau_{\uparrow\lambda}$ -open  $A$ ,

$$f(A \cap (\alpha, \beta)^{\tau_{\uparrow\lambda}}) = f(A) \cap (\alpha, \beta)^{\sigma_{\uparrow\lambda}} \in \sigma_{\uparrow\lambda+1};$$

and for any  $\sigma_{\uparrow\lambda}$ -open  $B$ ,

$$f^{-1}(B \cap (\alpha, \beta)^{\sigma_{\uparrow\lambda}}) = f^{-1}(B) \cap (\alpha, \beta)^{\tau_{\uparrow\lambda}} \in \tau_{\uparrow\lambda+1};$$

so that  $f$  is  $(\tau_{\uparrow\lambda+1}, \sigma_{\uparrow\lambda+1})$ -continuous and open. Clearly it is also pointwise discrete. The case for limit  $\lambda$  follows from [15, Lemma 5.8].  $\square$

As mentioned in the proof of 4.2.3, Lemma 4.2.2 implies that d-maps are rank-preserving. Also, it follows from 4.2.3 that if the rank of  $(X, \tau)$  is  $\Theta$ , then

$$\rho_\tau : (X, \tau_{\uparrow\lambda}) \rightarrow (\Theta, \mathcal{I}_\lambda)$$

is a d-map. The main feature of d-maps is as follows:

**Lemma 4.3.** *GLP<sub>1</sub> is complete with respect to  $(X, \tau)$  if, and only if, for any finite, converse-wellfounded tree  $T$ , there exists a  $\tau$ -open subspace  $S$  of  $X$  and a d-map  $f : (S, \tau) \rightarrow T$ .*

*Proof.* That completeness follows from the existence of d-maps is independently due to M. Abashidze and A. Blass (see [1] or [12]). In our framework, it immediately follows from Proposition 2.2 and Lemma 4.2.4. We prove the converse. Suppose GLP<sub>1</sub> is complete with respect to  $(X, \tau)$ , where  $\tau$ . Let  $(T, <)$  be a finite, converse wellfounded tree. We define from  $T$  a modal formula  $\varphi$  consistent with GLP<sub>1</sub>. Let  $\{p_t : t \in T\}$  be a set of distinct propositional variables and  $r$  be the root of  $T$ . Set

$$\begin{aligned} \varphi = & p_r \wedge \bigwedge_{s, t \in T; s \neq r} \neg p_s \wedge \left( \bigwedge_{t \in T; t \neq r} \Diamond p_t \right) \wedge \Box \left( \bigvee_{t \in T} p_t \right) \wedge \Box (\neg p_r) \\ & \wedge \left( \bigwedge_{s, t \in T; s \neq t} \Box (p_s \rightarrow \neg p_t) \right) \wedge \left( \bigwedge_{s < t} \Box (p_s \rightarrow \Diamond p_t) \right) \\ & \wedge \left( \bigwedge_{s \not< t} \Box (p_s \rightarrow \neg \Diamond p_t) \right) \wedge \Box \left( p_t \rightarrow \Box \bigvee_{t < s} p_s \right). \end{aligned}$$

Clearly, there is a Kripke model based on  $T$  where  $\varphi$  is true in  $r$ ; namely, any one where each  $p_t$  holds only in  $t$ . Hence,  $\varphi$  is consistent with GLP<sub>1</sub>, whereby it is satisfiable in  $X$ . Fix a valuation over  $X$  and a point  $x_r \in X$  such that  $x_r \models \varphi$ . Thus,  $x_r$  satisfies  $p_r$  and  $\bigwedge_{s, t \in T; s \neq r} \neg p_s$  and  $x_r$  is a limit point of points satisfying each of  $p_t$ , for  $t \neq r$ . Moreover, by each of the conjuncts above:

- i. there is a punctured neighborhood of  $x_r$  where each point satisfies  $p_t$  for some  $t \in T$ ;
- ii. there is a punctured neighborhood of  $x_r$  where no point satisfies  $p_r$ ;
- iii. for each pair of distinct  $s, t \in T$ , there is an punctured neighborhood of  $x_r$  of points satisfying at most one of  $p_s$  and  $p_t$ ;
- iv. for each pair of distinct  $s, t \in T$  with  $s < t$ , there is an punctured neighborhood of  $x_r$  where all points satisfying  $p_s$  are limits of points satisfying  $p_t$ ;
- v. for each pair of distinct  $s, t \in T$  with  $s \not< t$ , there is an punctured neighborhood of  $x_r$  where all points satisfying  $p_s$  are not limits of points satisfying  $p_t$ ; and
- vi. there is a punctured neighborhood of  $x_r$  where whenever a point  $x$  satisfies  $p_t$ , then there is a punctured neighborhood of  $x$  where each points satisfies one of  $p_s$ , with  $s < t$ .

Let  $S$  be the intersection of all those finitely many open sets. Clearly,  $x \models p_t$  and  $t \neq s$  together imply  $x \not\models p_s$ . We define  $f: S \rightarrow T$  by

$$f(x) = t \text{ if, and only if, } x \models p_t.$$

We claim  $f$  is a d-map. Let  $A_t$  be an open subset of  $T$  of the form

$$\{s \in T: t \leq s\},$$

so that

$$f^{-1}(A_t) = \{x \in S: x \models p_s \wedge t \leq s\}.$$

This clearly equals  $S$  if  $t = r$ . Otherwise, for each  $x \in S$  with  $x \models p_s$  and  $t \leq s$ , there is an open neighborhood  $U$  of  $x$  where each point satisfies  $p_u$  for some  $u > s$ . But  $t < u$ , whence  $U \subset f^{-1}(A_t)$ . This implies that  $f^{-1}(A_t)$  is open, and so  $f$  is continuous.

Conversely, suppose  $U \subset S$  is open,  $x \in U$  is such that  $x \models p_t$ , and  $t < s$ . Then  $x$  is a limit of points satisfying  $p_s$ , so that

$$\{y \in S: y \models p_s\} \cap U \neq \emptyset,$$

whence  $f(U) \cap A_s \neq \emptyset$ . Hence,  $f$  is open. Finally if  $t \in T$ , then  $f^{-1}(t)$  is discrete, for  $t$  is the image of points satisfying  $p_t$  and no point in  $S$  can satisfy  $p_s \wedge \Diamond p_s$  for any  $s$ . Therefore,  $f$  is a d-map.  $\square$

In fact, the following is a direct consequence of the proof of Lemma 4.3:

**Corollary 4.4.** *If  $\text{GLP}_1$  is complete with respect to  $(X, \tau, \mathcal{A})$ , then for any finite, converse-wellfounded tree  $T$ , there exists a  $\tau$ -open subspace  $S$  of  $X$  and a d-map  $f: (S, \tau) \rightarrow T$  with the property that the preimage of any node belongs to  $\mathcal{A}$ .*

Hence, the need to check whether a given space  $\mathfrak{X}$  satisfies a formula is replaced by the definition of a suitable mapping between  $\mathfrak{X}$  and some other space which is known to do so. Moreover, by Lemma 4.2.2, this result can be relativized to ambiances (condition (2.1) is preserved). In practice, polymodal analogs of Lemma 4.3 do not even require us to use full d-maps, but rather a weaker form of embeddings.<sup>3</sup> This is outlined as follows.

**Definition 4.5** (J-frame). A finite polymodal Kripke frame  $\mathfrak{F} = (W, \{\prec_n\}_{n < \omega})$  is called a *J-frame* if each relation is transitive and conversely wellfounded and it satisfies the following two conditions:

- (I) For all  $x, y \in W$  and all  $m < n$ :  $x \prec_n y$  implies that for all  $z \in W$ :  $x \prec_m z$  if, and only if  $y \prec_m z$ .
- (J) For all  $x, y, z \in W$  and all  $m < n$ : if  $x \prec_m y$  and  $y \prec_n z$ , then  $x \prec_m z$ .

We call a J-frame a *J<sub>n</sub>-frame* if all binary relations past the  $n$ th one are empty.

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<sup>3</sup>The technology was developed in [7].

Let  $(T, <_0, \dots, <_N)$  be a frame. Denote by  $E_n$  the reflexive, symmetric, and transitive closure of  $\bigcup_{n \leq k < \omega} R_k$ . The equivalence classes under  $E_n$  are called *n-planes*. A natural order is defined on the set of  $(n+1)$ -planes:

$$\alpha \prec \beta \text{ if, and only if, } x <_n y \text{ for some } x \in \alpha, y \in \beta$$

We say that a J-frame is a *J-tree* if for all  $n$ , the  $(n+1)$ -planes contained in each  $n$ -plane form a tree under  $R_n$  and if whenever  $\alpha < \beta$  for two  $(n+1)$ -planes  $\alpha, \beta$ , we have  $x R_n y$  for all  $x \in \alpha$  and  $y \in \beta$ . This means that each treelike  $J_n$ -frame can be thought of as a tree, each of whose nodes are  $J_{n-1}$ -trees.

**Definition 4.6 (J-map).** Let  $(T, \sigma_0, \dots, \sigma_n)$  be a  $J_n$ -tree and  $(X, \tau_0, \dots, \tau_n)$  be a  $\text{GLP}_n$ -space. We say that a function  $f : X \rightarrow T$  is a *J<sub>n</sub>-map* if

- (j<sub>1</sub>)  $f : (X, \tau_n) \rightarrow (T, \sigma_n)$  is a d-map;
- (j<sub>2</sub>)  $f : (X, \tau_k) \rightarrow (T, \sigma_k)$  is open for each  $k$ ;
- (j<sub>3</sub>)  $f^{-1}(\ll (x)), f^{-1}(\{x\} \cup \ll (x)) \in \tau_k$  for each  $k < n$  and each hereditary  $(k+1)$ -root  $x$ ;
- (j<sub>4</sub>)  $f^{-1}x$  is a  $\tau_k$ -discrete subspace for each  $k < n$  and each hereditary  $(k+1)$ -root  $x$ .

The following analog of the notion of J-maps for polytopologies was noted in [15], as it allows us to focus only on finitely-many modalities at a time.

**Definition 4.7.** Let  $(X, \{\mathcal{T}_i\}_{i < \Lambda})$  be a polytopological space,  $\vec{\sigma}$  an increasing  $n$ -sequence of nonzero ordinals, and  $T$  a  $J_n$ -tree. A function  $f : X \rightarrow T$  is a  $\vec{\sigma}$ -map if

$$f : (X, \mathcal{T}_{\sigma_0}, \mathcal{T}_{\sigma_1}, \dots, \mathcal{T}_{\sigma_n}) \rightarrow T$$

is a J-map.

This observation was made in [7]:

**Lemma 4.8.** *If  $f : Y \rightarrow Z$  is a  $J_n$ -map and  $g : X \rightarrow Y$  is a d-map, then  $f \circ g : X \rightarrow Z$  is a  $J_n$ -map.*

The following result is essentially proved in [9]:

**Lemma 4.9.** *For each  $\mathcal{L}_n$ -formula  $\varphi$ , there exists a  $J_n$ -tree  $T$  such that if  $\mathfrak{X}$  is a  $\text{GLP}_n$ -ambience and  $f : \mathfrak{X} \rightarrow T$  is a surjective  $J_n$ -map such that the preimage of each point is an element of the algebra in  $\mathfrak{X}$ , then  $\mathfrak{X} \models \varphi$  implies  $\text{GLP}_n \vdash \varphi$ .*

We call the tree obtained in Lemma 4.9 the *canonical tree* for  $\varphi$ .

## 4.2 Icard Spaces Revisited

In this section, we prove that our definition of the generalized Icard topologies is equivalent to that of [3]. This is necessary, as we will rely heavily on results therein during the proof of the completeness theorem.

**Definition 4.10** (Hyperlogarithms and  $-$ -exponentials, [17]).

1. The *hyperlogarithms*  $\{\ell^\xi\}_{\xi \in \text{Ord}}$  are the unique family of pointwise maximal initial<sup>4</sup> functions that satisfy:
  - (a)  $\ell^1 = \ell$ , and
  - (b)  $\ell^{\alpha+\beta} = \ell^\beta \ell^\alpha$ .
2. Let the function  $e$  be defined by  $\xi \mapsto -1 + \omega^\xi$ . The *hyperexponentials*  $\{e^\zeta\}_{\zeta \in \text{Ord}}$  are the unique pointwise minimal family of normal functions that satisfy
  - (a)  $e^1 = e$ , and
  - (b)  $e^{\alpha+\beta} = e^\alpha e^\beta$  for all  $\alpha$  and  $\beta$ .

One can verify by induction that the sequence  $\{\ell^\xi \gamma : \xi \in \text{Ord}\}$  is non-increasing for any ordinal  $\gamma$ . If we set  $e^0$  to be the identity function and  $e^\xi 0 = 0$  for all  $\xi$ , then one can also describe hyperexponentials recursively by condition 4.10.2b, together with the following normality clause:

$$\text{for any } \xi \text{ and any limit } \lambda : e^\xi \lambda = \lim_{\eta \rightarrow \lambda} e^\xi \eta; \quad (4.1)$$

and the following fixed-point clause:

$$\text{for any } \xi \text{ and any limit } \lambda : e^\lambda(\xi + 1) = \lim_{\eta \rightarrow \lambda} e^\eta(e^\lambda \xi + 1). \quad (4.2)$$

Indeed, condition (4.1) states that the hyperexponentials are normal functions and condition (4.2) states that each ordinal in the range of a hyperexponential is a fixed point of all previous hyperexponentials. Consequently, we see that the hyperexponential family refines the Veblen hierarchy. We mention some more properties of hyperlogarithms and -exponentials.

**Lemma 4.11** ([15, 17]).

1. If  $\xi$  and  $\delta$  are nonzero, then  $\ell^\xi(\gamma + \delta) = \ell^\xi \delta$ ; if  $\gamma < \delta$  as well, then  $\ell^\xi(-\gamma + \delta) = \ell^\xi \delta$ . Moreover, if  $1 < \xi$ , then  $\ell^\xi(\gamma \delta) = \ell^\xi \delta$ ;
2. If  $\xi < \zeta$ , then  $\ell^\xi e^\zeta = e^{-\xi+\zeta}$  and  $\ell^\zeta e^\xi = \ell^{-\xi+\zeta}$ . Furthermore, if  $\alpha < e^\xi \beta$ , then  $\ell^\xi \alpha < \beta$ .

*Sketch of 1.* That  $\ell^\xi(\gamma + \delta) = \ell^\xi \delta$  is proved by induction on  $\xi$  using 4.10.1b. From this follows that if  $\gamma < \delta$ , then

$$\ell^\xi(\delta) = \ell^\xi(\gamma + (-\gamma + \delta)) = \ell^\xi(-\gamma + \delta).$$

Finally, one can prove by induction that  $\ell(\gamma \delta) = L\gamma + \ell\delta$ , so that if  $1 < \xi$ , then

$$\ell^\xi(\gamma \delta) = \ell^{-1+\xi} \ell(\gamma \delta) = \ell^{-1+\xi}(L\gamma + \ell\delta) = \ell^{-1+\xi} \ell\delta = \ell^\xi \delta,$$

as desired. □

We now give an alternative characterization of topologies  $\tau_{\uparrow\lambda}$  and their rank functions:

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<sup>4</sup>That is, sending initial segments of  $\text{Ord}$  onto initial segments of  $\text{Ord}$ .

**Lemma 4.12.** *Let  $(X, \tau)$  be a scattered space of rank  $\Theta$ .*

1. *The topologies  $\tau_{\uparrow\lambda}$  are computed as follows:*

- $\tau_{\uparrow 0}$  is equal to  $\tau$
- $\tau_{\uparrow\lambda}$  generated by  $\tau$  and all sets of the form

$$(\alpha, \beta]_{\xi}^{\tau} := \{x \in X : \alpha < \ell^{\xi} \rho_{\tau} x \leq \beta\},$$

*for some  $-1 \leq \alpha < \beta \leq \Theta$  and some  $\xi < \lambda$ .*

2. *If  $(X, \tau)$  is a scattered space, then  $\rho_{\tau_{\uparrow\lambda}} = \ell^{\lambda} \circ \rho_{\tau}$ . In particular, the rank of  $\mathcal{I}_{\lambda}$  is  $\ell^{\lambda}$ .*

Sets of the form  $[\alpha, \beta]_{\xi}^{\tau}$ ,  $[\alpha, \beta)_{\xi}^{\tau}$ , and  $(\alpha, \beta)_{\xi}^{\tau}$  are defined in the obvious way. In particular, note that  $(\alpha, \beta)_0^{\tau} = (\alpha, \beta)^{\tau}$ .

*Proof.* The second claim follows from Lemma 4.2.1 and Lemma 4.2.3. We use this to prove the first claim by induction. Suppose  $\tau_{\uparrow\lambda}$  is generated by  $\tau$  and all sets of the form

$$(\alpha, \beta]_{\xi}^{\tau} := \{x \in X : \alpha < \ell^{\xi} \rho_{\tau} x \leq \beta\},$$

for  $\xi < \lambda$ . By definition,  $\tau_{\uparrow(\lambda+1)} = (\tau_{\uparrow\lambda})_{\uparrow 1}$  is generated by  $\tau_{\uparrow\lambda}$  and all sets of the form

$$(\alpha, \beta)^{\tau_{\uparrow\lambda}} := \{x \in X : \alpha < \rho_{\tau_{\uparrow\lambda}} x < \beta\},$$

but  $\rho_{\tau_{\uparrow\lambda}} = \ell^{\lambda} \circ \rho_{\lambda}$  by induction hypothesis. So  $(\tau_{\uparrow\lambda})_{\uparrow 1}$  is generated by all sets of the form

$$(\alpha, \beta]_{\xi}^{\tau} := \{x \in X : \alpha < \ell^{\xi} \rho_{\tau} x \leq \beta\},$$

for  $\xi < \lambda + 1$ . The limit case is immediate. □

As a consequence, we obtain:

**Lemma 4.13.** *Hyperlogarithms  $\ell^{\xi} : (\Theta, \mathcal{I}_{\xi+\zeta}) \rightarrow (\Theta, \mathcal{I}_{\zeta})$  are  $d$ -maps.*

*Proof.* See [15]. □

We will make use of the following Lemma from [3], which allows us to separate a point from other points of equal or greater rank inside generalized Icard topologies:

**Lemma 4.14.** *Let  $(X, \tau)$  be a scattered space and  $\lambda$  be an ordinal. Any  $x$  in  $(X, \tau_{\uparrow\lambda})$  has a  $\lambda$ -neighborhood  $U$  such that whenever  $x \neq y \in U$ ,  $\ell^{\lambda} \rho_0 y < \ell^{\lambda} \rho_0 x$ .*

We will also make use of some particular neighborhood bases of some points in Icard spaces:

**Lemma 4.15** ([3, Lemma 5.10]). *Let  $1 < \lambda$  be an additively indecomposable ordinal and  $x \in X$  be such that  $\rho_{\tau} x = e^{\lambda} \Theta > 0$ . Then for any  $\tau_{\uparrow\lambda}$ -neighborhood  $V$  of  $x$ , there exist*

- a set  $U \in \tau$ , and
- ordinals  $\eta < e^{\lambda} \Theta$  and  $\zeta < \lambda$ ,

*such that  $V$  contains the set  $U \cap (\eta, e^{\lambda} \Theta]_{\zeta}^{\tau}$ .*

In general, the conclusion of Lemma 4.15 does not hold for arbitrary  $\mathcal{I}_\lambda$ -limit points. The proof of the embedding lemma will make use of the following ‘copying construction’:

**Lemma 4.16** (see [15]). *Suppose  $f: (X, \tau) \rightarrow (Y, \sigma)$  is a  $d$ -map and  $\mathfrak{Y} = (Y, \{\mathcal{S}_\iota\}_{\iota < \Lambda})$  is a BG-space over  $(Y, \sigma)$ . Then, there exists a BG-space  $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  over  $(X, \tau)$  such that*

$$f: (X, \mathcal{T}_\iota) \rightarrow (Y, \mathcal{S}_\iota)$$

*is a  $d$ -map for each  $\iota$ .*

A consequence is the following:

**Corollary 4.17** (pullback). *Suppose  $\mathfrak{Y}$  is a maximal (resp. minimal)  $\text{GLP}_\Lambda$ -ambiance over  $(Y, \sigma)$  and*

$$f: (X, \tau) \rightarrow (Y, \sigma)$$

*is a  $d$ -map. Then, there exists a maximal (resp. minimal)  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  over  $(X, \tau)$  such that*

$$f: (X, \mathcal{T}_\iota) \rightarrow (Y, \mathcal{S}_\iota)$$

*is a  $d$ -map for each  $\iota$ .*

*Proof.* First suppose  $\mathfrak{Y}$  is maximal. We apply Lemma 4.16 to obtain a polytopology  $(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda})$  over  $(X, \tau)$  and equip it with the algebra  $\wp(X)$ . By Lemma 3.4, the resulting ambiance satisfies (2.1).

Now suppose  $\mathfrak{Y} = (Y, \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  is minimal. By definition,  $\{\mathcal{S}_\iota\}_{\iota < \Lambda}$  are consecutive Icard topologies starting at, say  $\sigma_{\uparrow\zeta}$ . We define a polytopology on  $X$  to consist of consecutive Icard topologies starting at  $\tau_{\uparrow\zeta}$ . It follows directly from Lemma 4.2.3 that

$$f: (X, \mathcal{T}_\iota) \rightarrow (Y, \mathcal{S}_\iota)$$

for each  $\iota$ , and by Lemma 4.2.2 we have that

$$f^{-1}\mathcal{B} := \{f^{-1}B : B \in \mathcal{B}\}$$

is a Boolean algebra satisfying (2.1). □

Corollary 4.17 and Lemma 4.18 below readily imply our main results.

**Lemma 4.18** (Embedding Lemma). *Let  $(T, <_0, <_1, \dots, <_n)$  be a finite  $J_n$ -tree with root  $r$  and  $\vec{\sigma}$  be an increasing  $n$ -sequence of nonzero ordinals with supremum below a multiplicatively indecomposable ordinal  $\Lambda$ . Then, there exist*

- *an ordinal  $\text{GLP}_\Lambda$ -ambiance  $([1, \Theta], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  over  $([1, \Theta], \mathcal{I}_\Lambda)$  such that  $\Theta < e^\Lambda 1$ ; and*
- *a surjective  $\vec{\sigma}$ -map  $f: [1, \Theta] \rightarrow T$  such that  $f^{-1}(r) = \{\Theta\}$ .*

*Moreover,  $\mathcal{A}$  can be taken to be minimal or maximal.*

*Promise of proof.* The Lemma is proved in Section 4.3 by making use of an auxiliary *Product Lemma* that is proved in Section 4.4. □

This gives:

**Theorem 4.19.** *Let  $\Lambda$  be any ordinal and  $(X, \tau)$  be any scattered space such that  $e^{e^{2\Lambda} \cdot 2} 1 \leq ht(X)$ . Then, there exist a Boolean algebra  $\mathcal{A}$  over  $X$  such that*

$$\mathcal{L}(X, \{\tau_{\uparrow\Lambda+\lambda}\}_{\lambda < \Lambda}, \mathcal{A}) = \text{GLP}_\Lambda.$$

*Proof.* Without loss of generality, assume  $\Lambda$  is multiplicatively indecomposable; otherwise substitute  $\bar{\Lambda} := \omega^{\omega^\Lambda}$  for it in the proof (hence, the ' $e^2\Lambda$ ' as the bound's exponent in the statement of the theorem). Suppose  $\text{GLP}_\Lambda \not\models \varphi_0$ . Let  $\vec{\sigma} = (\sigma_0, \dots, \sigma_N)$  be the modalities appearing in  $\varphi_0$ . Then  $\text{GLP}_N \not\models \varphi_0^\sigma$  by Lemma 2.7. We use Lemmata 4.9 and 4.18 to find the canonical tree  $T$  for  $\varphi$ , a minimal ordinal  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{X}_\varphi$  over  $([1, \Theta], \mathcal{I}_\Lambda)$  and a surjective  $\vec{\sigma}$ -map  $f : [1, \Theta] \rightarrow T$  respecting the algebra of  $\mathfrak{X}_\varphi$ , so that  $\mathfrak{X}_\varphi \not\models \varphi$ . We repeat this procedure for each formula  $\varphi \in \mathcal{L}_\Lambda$  and take  $\mathfrak{X} = \bigsqcup_{\varphi \in \mathcal{L}_\Lambda} \mathfrak{X}_\varphi$ .

Now let  $(X, \tau)$  be an arbitrary scattered space of rank  $\geq e^{\Lambda \cdot 2} 1$ , so that  $X_{\uparrow\Lambda}$  has rank  $\geq e^\Lambda 1 = e^{1+\Lambda} 1$ . Use Corollary 4.17 to build a minimal  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{X}' = (X, \{\tau_\lambda\}_\lambda)$  over  $(X, \tau)$  such that

$$\rho_\tau : (X, \tau_\lambda) \rightarrow (S, \mathcal{T}_\lambda)$$

is a d-map for each  $\lambda$  and some open subspace  $S$  of  $[1, \Theta]$ . By Lemmata 4.9 and 4.8, any formula consistent with  $\text{GLP}_\Lambda$  is satisfiable in  $\mathfrak{X}'$ .  $\square$

Notice that the ambiances constructed in the proof of Theorem 4.19 are built on top of a space of rank  $\geq e^\Lambda 1$ . By Lemma 4.3, this is the best possible.

**Theorem 4.20.** *Let  $\Lambda$  be any ordinal and  $(X, \tau)$  be a scattered space such that  $e^{e^{2\Lambda} \cdot 2} 1 \leq ht(X)$ . Then, there exist a BG-polytopology  $\{\mathcal{T}_\iota\}_{\iota < \Lambda}$  over  $\tau_{\uparrow\Lambda}$  such that*

$$\mathcal{L}(X, \{\mathcal{T}_\iota\}_{\iota < \Lambda}) = \text{GLP}_\Lambda.$$

*Proof.* We repeat the proof of Theorem 4.19, with the only exception that we that build the ordinal ambiance  $\mathfrak{X} = ([1, \Theta], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \wp([1, \Theta]))$  using maximal algebras only. We omit the details.  $\square$

### 4.3 The embedding lemma

As anticipated, the proof of Lemma 4.18 relies on the following technical lemma whose proof will be provided in the next subsection:

**Lemma 4.21** (Product Lemma). *Assume  $\varsigma$  is a multiplicatively indecomposable ordinal,  $([1, \kappa], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  and  $([1, \lambda], \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  are maximal (resp. minimal) ordinal  $\text{GLP}_\Lambda$ -ambiances over  $\mathcal{I}_\varsigma$ , and  $\{\kappa_0, \dots, \kappa_n\}$  is a finite subset of  $[1, \kappa]$ . Then, there exist:*

1. *An ordinal  $\Theta < e^\varsigma(\ell^\varsigma[1, \kappa]) \cdot \lambda$  and a partition of  $\Theta$  into two  $\mathcal{I}_{\varsigma+1}$ -clopen sets  $X_\uparrow$  and  $X_\downarrow$ .*
2. *A maximal (resp. minimal)  $\text{GLP}_\Lambda$ -ambiance  $([1, \Theta], \{\mathcal{R}_\iota\}_{\iota < \Lambda}, \mathcal{C})$  over  $\mathcal{I}_\varsigma$ .*
3. *Functions  $\pi_0 : [1, \Theta] \rightarrow [1, \kappa]$  and  $\pi_1 : [1, \Theta] \rightarrow [1, \lambda]$  such that:*
  - $\pi_0 : (X_\downarrow, \mathcal{R}_\iota) \rightarrow ([1, \kappa], \mathcal{T}_\iota)$  is a d-map for each  $\iota$ ;
  - $\pi_1 : (X_\uparrow, \mathcal{R}_\iota) \rightarrow ([1, \lambda], \mathcal{S}_\iota)$  is a d-map for each  $\iota > 0$ ;



- $X_{\uparrow} \subset d\pi_0^{-1}(\kappa_i)$  for any  $i < n$ .

**Remark 4.22.** In particular, if  $\kappa, \lambda < e^{\varsigma}\alpha$ , then the ordinal  $\Theta$  given by Lemma 4.21 satisfies  $\Theta < e^{\varsigma}\alpha$ .

**Remark 4.23.** In order to prove Lemma 4.21, one has to deal with many difficulties that are not present if one only wishes to consider countable  $\Lambda$ . In particular, the nonexistence of d-maps from higher Icard topologies onto the lower ones (see [3, Theorem 6.2]) forces us to consider the case  $\varsigma \neq 1$ , which requires substantially more work. In particular, it is not clear whether a general algebraic construction can be attained, as the proof depends deeply on the arithmetical structure of the ordinals.

Moreover, difficulties arise when proving that the projections  $\pi_0$  and  $\pi_1$  have the required properties and many computations are involved. In particular, one has to show that they preserve a bit more structure than what is stated in the lemma in order to ensure that enough information about the ambiances in  $[1, \kappa]$  and  $[1, \lambda]$  can be recovered.

*Proof of Lemma 4.18.* We prove this by induction on  $n$ . The case when  $n = 0$  is a consequence of (the proof of the) main theorem of [3]. Hence, we assume  $1 < n$  (so that, in particular  $1 < \Lambda$ ) and that the result holds for all numbers  $< n$ . We proceed by an auxiliary induction on the height of  $<_0$ ,  $\text{hgt}_0(T)$ .

**Case I:**  $\text{hgt}_0(T) = 0$ . We have that  $<_0 = \emptyset$ . Let

$$\begin{aligned}\vec{\sigma}^\circ &:= -\sigma_0 + \vec{\sigma} \upharpoonright \{1, \dots, n\} \\ &= (-\sigma_0 + \sigma_1, \dots, -\sigma_0 + \sigma_n).\end{aligned}$$

By induction hypothesis, there is a  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{X} = ([1, \Xi], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  over  $([1, \Xi], \mathcal{I}_\Lambda)$  and an onto  $\vec{\sigma}^\circ$ -map

$$f: ([1, \Xi], \mathcal{T}_{-\sigma_0+\sigma_n}, \dots, \mathcal{T}_{-\sigma_0+\sigma_n}) \rightarrow (T, <_1, \dots, <_n)$$

such that  $\Xi < e^\Lambda 1$ , and  $f^{-1}(r) = \{\Xi\}$ . We apply Corollary 4.17 to find a  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{Y} = ([1, e^{\sigma_0}\Xi], \{\mathcal{T}'_\iota\}'_{\iota < \Lambda}, \mathcal{A}')$  such that

$$\ell^{\sigma_0}: ([1, e^{\sigma_0}\Xi], \mathcal{T}'_{\sigma_0}, \dots, \mathcal{T}'_{\sigma_n}) \rightarrow ([1, \Xi], \mathcal{T}_{-\sigma_0+\sigma_n}, \dots, \mathcal{T}_{-\sigma_0+\sigma_n})$$

is a d-map. Note that  $\Theta := e^{\sigma_0}\Xi < e^\Lambda 1$ . By Lemma 4.8,

$$f \circ \ell^{\sigma_0}: ([1, e^{\sigma_0}\Xi], \mathcal{T}'_{\sigma_0}, \dots, \mathcal{T}'_{\sigma_n}) \rightarrow (T, <_1, \dots, <_n)$$

is a  $\vec{\sigma}^\circ$ -map. In fact, it is a  $\vec{\sigma}$ -map: condition  $(j_1)$  is given by induction, as are conditions  $(j_2)$ – $(j_4)$  for  $k \geq 0$ . Moreover,  $(j_2)$  is satisfied trivially for  $k = 0$  since the topology induced by  $R_0$  is discrete;  $(j_3)$  and  $(j_4)$  hold because the only hereditary 1-root is the root  $r$  whose preimage is  $e^{\sigma_0}\Xi$ .

**Case II:**  $0 < n := \text{hgt}_0(T)$  and  $\Lambda$  is additively indecomposable. Let  $r_1, \dots, r_m$  be all  $<_0$ -successors of  $r$  that are hereditary 1-roots and  $\ll_0(r_i)$  denote the generated subtrees. Also let  $\ll_1(r)$  denote the subtree consisting of all nodes that are  $<_0$ -incomparable with  $r$  (i.e., the  $<_0$ -roots). By induction hypothesis, there exist  $\text{GLP}_\Lambda$ -ambiances

$$\mathfrak{X}_i = ([1, \kappa_i], \{\mathcal{T}_\iota\}_{\iota < \Lambda}^i, \mathcal{A}^i)$$

over  $([1, \kappa_i], \mathcal{I}_1)$  and onto  $\vec{\sigma}$ -maps  $f_i: \mathfrak{X}_i \rightarrow \ll_0 (r_i)$  for  $0 < i$ . Let

$$\kappa = \kappa_1 + \dots + \kappa_m$$

and  $\mathfrak{X} = ([1, \kappa], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  be the topological sum. We also denote by  $f^*$  the sum of the functions  $f_i$ . Define an ambiance  $\mathfrak{Y} = ([1, \lambda], \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  exactly as in Case I in such a way that there is a  $\vec{\sigma}$ -map

$$f_0: \mathfrak{Y} \rightarrow (\ll_1 (r), <_0, \dots, <_n).$$

Let

- functions  $\pi_0, \pi_1$ ,
- a  $\text{GLP}_\Lambda$ -ambiance  $\mathfrak{Z} = ([1, \Theta], \{\mathcal{R}_\iota\}_{\iota < \Lambda}, \mathcal{C})$ , and
- a clopen partition  $\{X_\uparrow, X_\downarrow\}$  of  $[1, \Theta]$ ,

be as described in the statement of Lemma 4.21. Moreover, assume that  $X_\uparrow \subset d\pi_0^{-1}(\kappa_i)$  for any  $i < m$ . Define the function  $f: \mathfrak{Z} \rightarrow T$  as follows:

$$f(x) = \begin{cases} f^*(\pi_0(x)) & \text{if } x \in X_\downarrow \\ f_0(\pi_1(x)) & \text{if } x \in X_\uparrow \end{cases}$$

The projections

$$\pi_0: (X_\downarrow, \mathcal{R}_\iota) \rightarrow ([1, \kappa], \mathcal{T}_\iota)$$

and

$$\pi_1: (X_\uparrow, \mathcal{R}_\iota) \rightarrow ([1, \lambda], \mathcal{S}_\iota)$$

are d-maps and  $f^*$  and  $f_0$  are  $\vec{\sigma}$ -maps. This fact and the induction hypothesis yield condition  $(j_1)$ , as well as conditions  $(j_2)$ – $(j_4)$  for  $0 < i$ . We verify the remaining ones:

- ( $j_2$ ) Let  $U$  be  $\mathcal{R}_{\sigma_0}$ -open. If  $U \subset X_\downarrow$ , then  $f(U)$  is  $<_0$ -open, as  $f^* \circ \pi_0$  is a  $\vec{\sigma}$ -map. If  $U \cap X_\uparrow \neq \emptyset$ , then we claim  $f(U)$  is  $<_0$ -open in  $T$ . Indeed, since  $X_\uparrow \subset d\pi_0^{-1}(\kappa_i)$  for any  $i < m$ , then there are ordinals  $\xi_0, \dots, \xi_m \in U$  such that  $\pi_0(\xi_i) = \kappa_i$  for each  $i$ . But then  $\pi_0(U \cap X_\downarrow)$  contains a neighborhood  $U_i$  of each  $\kappa_i$  and by induction hypothesis,  $f^*(U_i) = \ll_0 (r_i)$ .
- ( $j_3$ ) Any hereditary 1-root  $x$  is either  $r$  or in some  $T_i$ . In the former case,  $f^{-1}(\ll_0 (r))$  and  $f^{-1}(\{r\} \cup \ll_0 (r)) \in \tau_0$  equal  $[1, \Theta)$  and  $[1, \Theta]$ , respectively. In the latter case, the result follows from the continuity of  $\pi_0$  and the fact that  $f^*$  is a  $\vec{\sigma}$ -map.
- ( $j_4$ ) Again,  $f^{-1}(r) = \Theta$  and for any hereditary 1-root  $x \neq r$ ,  $f^{-1}(x) = \pi_0^{-1}f^{*-1}(x)$  is discrete because  $f^{*-1}(x)$  is discrete and  $\pi_0$  is pointwise discrete and continuous.

Therefore,  $f$  is a  $\vec{\sigma}$ -map. □

#### 4.4 Proof of the Product Lemma

In this section, we prove Lemma 4.21. A slight change in notation will make the proof easier: instead of starting with an ordinal  $\kappa$  and a finite subset  $\{\kappa_1, \dots, \kappa_m\}$  of  $[1, \kappa]$ , we will start with a set of ordinals  $\{\kappa_1, \dots, \kappa_m\}$  and define  $\kappa$  as their sum.

So let  $\varsigma$  be a multiplicatively indecomposable ordinal (possibly equal to 1). Let  $\lambda, \kappa_1, \dots, \kappa_m$  be nonzero ordinals such that  $\kappa_1 \leq \dots \leq \kappa_m$  and write

$$\kappa = \kappa_1 + \dots + \kappa_m.$$

Note in particular that  $\kappa_0$  is undefined. The reason for this is that we will often speak of objects such as “ $\kappa_{\iota \bmod m}$ ,” and we would like  $\kappa_0$  to be the largest  $\kappa_\iota$ , as this will make notation a bit simpler. Here we are assuming that  $\omega \equiv 0 \bmod m$  by definition.

Assume  $([1, \kappa], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  and  $([1, \lambda], \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  are maximal (resp. minimal) ordinal GLP $_\Lambda$ -ambiances over  $\mathcal{I}_\varsigma$ . We let

$$\xi = \text{the unique ordinal such that } \xi = \ell^\varsigma[1, \kappa_m].$$

Necessarily  $\xi$  is a successor ordinal, and so we set  $\xi = \zeta + 1$ . As  $\kappa_m < e^\varsigma \xi$ , it follows from condition (4.2) that

$$e^\alpha(e^\varsigma \zeta + 1) \leq \kappa_m$$

for some greatest ordinal  $\alpha$ , which we will denote by  $\nu$ , i.e.,

$$e^\nu(e^\varsigma \zeta + 1) \leq \kappa_m < e^{\nu+1}(e^\varsigma \zeta + 1).$$

The proof of the Product Lemma is distributed among a series of lemmata and definitions throughout this section; the following should serve as an index that outlines its proof. Concretely, we need to define:

- An ordinal  $\Theta$  partitioned into sets  $X_\uparrow$  and  $X_\downarrow$ ; this is done in Definition 4.28.
- Projection functions  $\pi_0, \pi_1$ ; this is done in Definition 4.35.
- A polytopology  $\{\mathcal{R}_\iota\}_{\iota < \Lambda}$  and a Boolean algebra  $\mathcal{C}$ ; these are defined according as to whether we are interested in maximal or in minimal ambiances. The relevant results here are Lemma 4.50 and Lemma 4.60.

Moreover, we need to show:

- That  $X_\uparrow$  and  $X_\downarrow$  are  $\mathcal{I}_\varsigma$ -clopen, but this will be clear from their definition and the fact that  $\xi$  is a successor ordinal.
- That  $\Theta < e^\varsigma(\ell^\varsigma[1, \kappa]) \cdot \lambda$ ; this follows from Corollary 4.38.
- That  $\pi_0: (X_\downarrow, \mathcal{R}_\iota) \rightarrow ([1, \kappa], \mathcal{T}_\iota)$  is a d-map for each  $\iota$ ; this is immediate from Lemma 4.40.2 and Lemma 4.2.3.
- That  $\pi_1: (X_\uparrow, \mathcal{R}_\iota) \rightarrow ([1, \lambda], \mathcal{S}_\iota)$  is a d-map for each  $\iota > 0$ ; this follows from Lemma 4.46.

- That  $X_{\uparrow} \subset d\pi_0^{-1}(\kappa_i)$  for any  $i < n$ ; this is Lemma 4.41.

**Definition 4.24.** Let  $\iota < \varsigma$  be an ordinal and write

$$\iota = \omega \cdot \iota_0 + k$$

where  $k$  is finite. Let  $\mathbf{fi}(\iota) = k$  be the *finite part* of  $\iota$  and  $\mathbf{ml}(\iota)$  the *modified limit part* of  $\iota$  defined as

$$\mathbf{ml}(\iota) = \nu \cdot \iota_0. \quad (4.3)$$

It follows from Lemma 3.8 that  $\mathbf{ml}(\iota) = \nu \cdot \mathcal{L}\iota$ . We define the *characteristic sequence* for  $\varsigma$ ,  $\mathbf{cs}(\varsigma)$  as follows:

- If  $\varsigma = 1$ , then  $\mathbf{cs}(\varsigma)$  is the sequence  $\{\varsigma_\iota\}_{\iota < \omega}$  with constant value 0.
- If  $\varsigma$  is a limit ordinal, then  $\mathbf{cs}(\varsigma)$  is the sequence  $\{\varsigma_\iota\}_{\iota < \varsigma}$  given by  $\varsigma_\iota = \mathbf{ml}(\iota) + \mathbf{fi}(\iota)$ .

**Remark 4.25.** The case  $\varsigma = 1$  needs to be considered separately. It corresponds to the notion of d-products from [9]. In our context, the Product Lemma will be sufficient, but it is also possible to ignore the case  $\varsigma = 1$  and replace this lemma with the d-product construction. However, the case  $\varsigma = 1$  will be so much simpler that it will only make marginal appearances during the proof.

**Lemma 4.26.** *The sequence  $\mathbf{cs}(\varsigma)$  is cofinal in  $\varsigma$ .*

*Proof.* As  $\varsigma$  is multiplicatively indecomposable, we have that  $\varsigma_\iota < \varsigma$  for each  $\iota < \varsigma$ . Now,

$$\sup_{\iota \in \varsigma} \varsigma_\iota = \sup_{\iota \in \varsigma \cap \text{Lim}} \varsigma_\iota = \sup_{\iota \in \varsigma \cap \text{Lim}} \nu \cdot \mathcal{L}\iota = \varsigma$$

□

**Remark 4.27.** Suppose we dropped the assumption that  $\varsigma$  be multiplicatively indecomposable and assumed that it is additively indecomposable, but not multiplicatively indecomposable. Then the proof of Lemma 4.26 does not go through. However, it is still the case that  $\varsigma$  is a limit of points in  $\mathbf{cs}(\varsigma)$ . To see this, note that  $\varsigma$  must be a limit of limit ordinals and so it suffices to show that there exists a limit ordinal of the form  $\omega \cdot \alpha \leq \varsigma$  such that

$$\nu \cdot \alpha = \varsigma. \quad (4.4)$$

But this is straightforward: let  $\alpha$  be least such that  $\nu \cdot \alpha \geq \varsigma$ . This  $\alpha$  cannot be a successor ordinal by additive indecomposability, whence  $\alpha$  is limit and (4.4) holds.

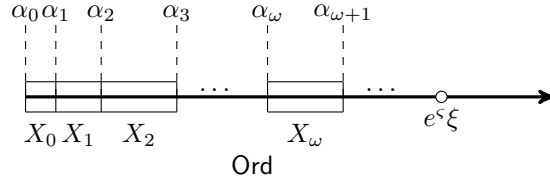
To maintain a bit more of notational simplicity, we will continue to treat  $\varsigma$  as multiplicatively indecomposable, although our construction can be easily modified around the weaker assumption: we only need to replace the sequence  $\mathbf{cs}(\varsigma)$  with the appropriate initial segment throughout. In Section 5, we will nonetheless state a result for which we will require to avoid the assumption that  $\varsigma$  be multiplicatively indecomposable. Consequently, we advise the careful reader to bear this in mind.

#### 4.4.1 The partition

**Definition 4.28**  $(\Theta, X_\uparrow, X_\downarrow)$ . For each  $\iota < \varsigma$ , we take  $\kappa_\iota$  to mean the unique  $\kappa_i$  such that  $\iota \equiv i \pmod m$ . In particular,  $\kappa_0 = \kappa_m \geq \kappa_i$  for any  $i$ . Set:

- $\alpha_0 = 1$ ;
- $\alpha_{\iota+1} = \beta_\iota + 1$ ;
- $\alpha_\iota = e^{\varsigma_\iota}(e^\varsigma \zeta + 1)$ , at limit stages; and
- $\beta_\iota = 1 + e^{\varsigma_\iota}(\kappa_0 + \kappa_\iota)$

Let  $X_\iota = [\alpha_\iota, \beta_\iota]$ ,  $Y_\iota = X_\iota \cap [0, \kappa_0]_{\varsigma_\iota}$ , and  $Z_\iota = X_\iota \setminus Y_\iota$ . We will soon prove (Lemma 4.32) that the family  $\{X_\iota : \iota < \varsigma\}$  partitions  $e^\varsigma \xi$  in the following way:



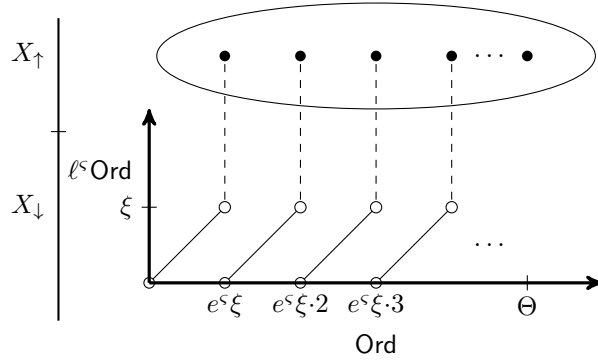
**Definition 4.29.** We define:

$$\Theta = \text{least } \alpha \text{ such that } d_{\mathcal{I}_\varsigma}^\xi[1, \alpha] \text{ has order-type } -1 + \lambda + 1.$$

We also set:

1.  $X_\downarrow = [1, \Theta] \cap [0, \xi)_\varsigma$ ;
2.  $X_\uparrow = [1, \Theta] \cap [\xi, \infty)_\varsigma$ .

See the following picture:



**Remark 4.30.** It follows from the definition that

$$\text{otyp}([1, \Theta] \cap [\xi, \infty)_\varsigma) = \text{otyp } X_\uparrow = -1 + \lambda + 1.$$

**Lemma 4.31.** The sets  $X_\iota$ ,  $Y_\iota$ , and  $Z_\iota$  are  $\mathcal{I}_\varsigma$ -clopen.

*Proof.* The sets  $X_\iota$  are clearly already  $\mathcal{I}_1$ -clopen. That the sets  $Y_\iota$  are  $\mathcal{I}_\varsigma$ -clopen follows from the fact that  $\varsigma_\iota < \varsigma$ ; consequently, so too are the sets  $Z_\iota$ .  $\square$

**Lemma 4.32.** *The sets  $\{X_\gamma : \gamma < \varsigma\}$  in Definition 4.28 form a partition of  $e^\varsigma \xi$ .*

*Proof.* From [3, Lemma 2.8] follows that

$$\sup_{\iota \in \varsigma \cap \text{Lim}} \alpha_\iota = e^\varsigma \xi,$$

so it suffices to show that

$$\lim_{\iota \rightarrow \gamma} \alpha_\iota = e^{\mathfrak{ml}(\gamma)}(e^\varsigma \zeta + 1) \text{ for each limit } \gamma. \quad (4.5)$$

We do this by induction. We write  $\gamma = \gamma^* + \omega^\rho$ . Note that it follows from (4.3) that if  $\alpha$  and  $\beta$  are limit ordinals, then  $\mathfrak{ml}(\alpha + \beta) = \mathfrak{ml}\alpha + \mathfrak{ml}\beta$ . Recall that the functions  $e^\iota$  are normal. Hence:

$$\begin{aligned} \lim_{\iota \rightarrow \gamma} \alpha_\iota &= \lim_{\iota \rightarrow \gamma} e^{\mathfrak{ml}(\iota) + \mathfrak{fi}(\iota)}(\kappa_0 \cdot 2 + 1) \\ &= \lim_{\iota \rightarrow \gamma} e^{\mathfrak{ml}(\iota) + \mathfrak{fi}(\iota) + 1}(\kappa_0) \\ &= \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\gamma^* + \iota) + \mathfrak{fi}(\gamma^* + \iota) + 1}(\kappa_0) \\ &= \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\gamma^*) + \mathfrak{ml}(\iota) + \mathfrak{fi}(\iota) + 1}(\kappa_0) \\ &= e^{\mathfrak{ml}(\gamma^*)} \left( \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota) + \mathfrak{fi}(\iota) + 1} \kappa_0 \right) \\ &= e^{\mathfrak{ml}(\gamma^*)} \left( \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)} \kappa_0 \right). \end{aligned} \quad (4.6)$$

Moreover,

$$\begin{aligned} \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota) + \nu}(e^\varsigma \zeta + 1) &\leq \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)} \kappa_0 \\ &\leq \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)} (e^{\nu+1}(e^\varsigma \zeta + 1)) \\ &= \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota) + \nu + 1}(e^\varsigma \zeta + 1) \\ &= \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota) + \nu}(e^\varsigma \zeta + 1), \end{aligned}$$

so that

$$\lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)}(e^\nu(e^\varsigma \zeta + 1)) = \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota) + \nu}(e^\varsigma \zeta + 1) = \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)} \kappa_0. \quad (4.7)$$

**Claim 4.33.** *For all ordinals  $\iota, \rho < \varsigma$ , the following equality holds:*

$$\lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)}(e^\nu(e^\varsigma \zeta + 1)) = e^{\nu \cdot \mathcal{L}\omega^\rho}(e^\varsigma \zeta + 1).$$

*Proof.* We use Lemma 3.8. We have that

$$\lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)}(e^\nu(e^\varsigma \zeta + 1)) = \lim_{\iota \rightarrow \omega^\rho} e^{\nu(\mathcal{L}\iota + 1)}(e^\varsigma \zeta + 1).$$

If  $\rho = 1$ , then,  $\mathcal{L}\iota$  is identically 0, so that

$$\lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)}(e^\nu(e^\varsigma \zeta + 1)) = e^\nu(e^\varsigma \zeta + 1) = e^{\nu \cdot \mathcal{L}\omega}(e^\varsigma \zeta + 1).$$

By (4.1),

$$\begin{aligned} e^{\nu \cdot \mathcal{L}\omega^\rho}(e^\zeta \zeta + 1) &= \lim_{\iota \rightarrow \nu \cdot \mathcal{L}\omega^\rho} \left( e^\iota (e^{\nu \cdot \mathcal{L}\omega^\rho} e^\zeta \zeta + 1) \right) \\ &= \lim_{\iota \rightarrow \nu \cdot \mathcal{L}\omega^\rho} (e^\iota (e^\zeta \zeta + 1)) \\ &= \lim_{\iota \rightarrow \omega^\rho} (e^{\nu \cdot \mathcal{L}\iota} (e^\zeta \zeta + 1)), \end{aligned}$$

so that if, on the other hand,  $1 < \rho$ , then  $\lim_{\iota \rightarrow \omega^\rho} (\mathcal{L}\iota + 1) = \lim_{\iota \rightarrow \omega^\rho} \mathcal{L}\iota = \mathcal{L}\omega^\rho$ , whence

$$\begin{aligned} \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)}(e^\nu(e^\zeta \zeta + 1)) &= \lim_{\iota \rightarrow \omega^\rho} e^{\nu(\mathcal{L}\iota + 1)}(e^\zeta \zeta + 1) \\ &= e^{\nu \cdot \mathcal{L}\omega^\rho}(e^\zeta \zeta + 1), \end{aligned}$$

as desired. This proves the claim.  $\square$

Finally, we have:

$$\begin{aligned} \lim_{\iota \rightarrow \gamma} \alpha_\iota &= e^{\mathfrak{ml}(\gamma^*)} \left( \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)} \kappa_0 \right), && \text{by (4.6),} \\ &= e^{\mathfrak{ml}(\gamma^*)} \left( \lim_{\iota \rightarrow \omega^\rho} e^{\mathfrak{ml}(\iota)} (e^\nu(e^\zeta \zeta + 1)) \right), && \text{by (4.7),} \\ &= e^{\mathfrak{ml}(\gamma^*)} \left( e^{\nu \cdot \mathcal{L}\omega^\rho} (e^\zeta \zeta + 1) \right), && \text{by the claim,} \\ &= e^{\mathfrak{ml}(\gamma^*) + \nu \cdot \mathcal{L}\omega^\rho} (e^\zeta \zeta + 1), \\ &= e^{\mathfrak{ml}(\gamma)} (e^\zeta \zeta + 1). \end{aligned}$$

Therefore, (4.5) holds for each limit  $\gamma$ .  $\square$

In order to define the projection  $\pi_0$ , we will refine  $X_\downarrow$  into a partition. This partition will be based on the sets  $X_\iota$ . Indeed, by Lemmata 4.31 and 4.32, the family  $\{X_\iota\}_{\iota < \varsigma}$  is already a clopen partition of a proper (unless  $\lambda = 1$ ) initial segment of  $X_\downarrow$ . The extension to all of  $X_\downarrow$  is then straightforward:

**Lemma 4.34.** *There exists an ordinal  $I$  and a partition of  $[1, \Theta] \cap [0, \xi)_\varsigma$  into  $\mathcal{I}_1$ -clopen sets  $\{X_\iota\}_{\iota < I}$  such that  $X_\iota$  is isomorphic to  $[\alpha_\rho, \beta_\rho]$  whenever  $\iota \equiv \rho \pmod \varsigma$ .*

*Proof.* We already defined the sets  $X_\iota$  for  $\iota < \varsigma$  and they form a partition of  $e^\varsigma \xi$  by Lemmata 4.31 and 4.32, so assume  $1 < \lambda$ . Let  $\iota \geq \varsigma$ , and write  $\iota = \varsigma \cdot \iota_0 + \rho$ , with  $\rho < \varsigma$ .

We consider three cases. First, suppose  $0 = \rho$ . Define  $\chi = \lim_{\eta \rightarrow \iota} \alpha_\eta$ . If  $\xi \leq \ell^\varsigma \chi$ , then set  $X_\iota = \chi + X_0$ ; else, set  $X_\iota = \chi + (-1 + X_0)$ . If, on the contrary,  $0 < \rho$ , then set  $X_\iota = \alpha_{\iota_0} + X_\rho$ . This determines a partition of  $X_\downarrow$ : if  $\alpha \in X_\downarrow$ , we can write  $\alpha$  in the form  $e^\varsigma \xi \cdot \alpha_0 + r$ . In this way,  $\alpha$  and  $r$  (resp.  $1 + r$ ) belong to isomorphic cells  $X_\iota$ .  $\square$

#### 4.4.2 Projections

**Definition 4.35** (Projections). We define the functions  $\pi_0$  and  $\pi_1$ :

( $\uparrow$ )  $\pi_1: X_\uparrow \rightarrow [1, \lambda]$  is defined by  $\alpha \mapsto 1 + \mathbf{otyp}_{X_\uparrow}(\alpha)$ .

( $\downarrow$ )  $\pi_0: X_\downarrow \rightarrow [1, \kappa]$  is defined by:

$$\pi_0(\alpha) = \begin{cases} 1 + \ell^{\varsigma_i} \alpha & \text{if } \ell^{\varsigma_i} \alpha \leq \kappa_0 \\ \kappa_0 + \dots + \kappa_{i-1} + 1 + (-(1 + \kappa_0) + \ell^{\varsigma_i} \alpha) & \text{otherwise;} \end{cases}$$

whenever  $i$  is the unique ordinal  $< m$  such that  $\alpha \in X_\iota$  for some  $\iota \equiv i \pmod m$ .

**Remark 4.36.** As we will soon see,  $\pi_0$  is a d-map.  $\pi_1$  is in general not a d-map when  $\varsigma$  is a limit. It is in general not open. For example, suppose  $\lambda = e^\varsigma 1$ . Then,  $\lambda$  is a limit in  $[1, \lambda]$ . We will have

$$\pi_1^{-1} \lambda = e^\varsigma \xi \cdot (e^\varsigma 1 + 1).$$

In particular,

$$(e^\varsigma \xi \cdot e^\varsigma 1, e^\varsigma \xi \cdot (e^\varsigma 1 + 1))_0$$

will be a neighborhood of  $\pi_1^{-1} \lambda$  whose image under  $\pi_1$  is not open. This problem was not present in the case  $\varsigma = 1$  as the image of nontrivial 0-intervals is nontrivial. It will complicate our proof slightly. However, as we will see, it can be fixed.

A problem that arises is that—unlike the case  $\varsigma = 1$ —we are not able to fully pull back the algebras from  $[1, \lambda]$ . Nonetheless, we shall observe that  $\pi_1$  preserves enough structure so that, if  $[1, \lambda]$  is equipped with a minimal or a maximal algebra, we will be able to recover through  $\pi_1$  enough information that will allow us to reconstruct an algebra for  $X_\uparrow$  that resembles the one in  $[1, \lambda]$ .

The following lemma computes  $X_\uparrow$  and  $\pi_1$ .

**Lemma 4.37.** *Let  $\alpha \in X_\uparrow$ .*

1. *If  $\varsigma = 1$ , then  $\alpha$  is of the form  $\kappa \cdot \omega \cdot \alpha_0$  and  $1 + \mathbf{otyp}_{X_\uparrow}(\alpha) = \alpha_0$ .*
2. *If  $\varsigma$  is a limit ordinal, then  $\alpha$  is of the form*

$$e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + e^1(e^\varsigma \xi + 1) \cdot \alpha_1 + (e^\varsigma \xi) \cdot \alpha_0. \quad (4.8)$$

*Let*

$$\rho = \ell^\varsigma \alpha_1$$

*and let  $\gamma_0$  be such that*

$$\gamma_0 + 1 = \alpha_0,$$

*if it exists. Then,*

- (a) *If  $\rho < \xi$  and  $\alpha_1 \in \mathbf{Lim}$ , then  $\gamma_0$  is well-defined, and*
- (b)  *$\pi_1 \alpha = 1 + \mathbf{otyp}_{X_\uparrow}(\alpha)$  is given by*

$$\pi_1 \alpha = \begin{cases} e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 + \gamma_0, & \text{if } \rho < \xi \text{ and } \alpha_1 \in \mathbf{Lim} \\ e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 + \alpha_0, & \text{otherwise.} \end{cases} \quad (4.9)$$



*Proof.* Item 1 is established by a straightforward induction on  $\alpha$ . For item 2, notice that

$$e^1(e^\varsigma\xi + 1) = \omega^{e^\varsigma\xi+1} = \omega^{e^\varsigma\xi}\omega = e^\varsigma\xi \cdot \omega,$$

so there are  $\omega$  points of nonzero  $\mathcal{I}_\varsigma$ -rank in  $[1, e^1(e^\varsigma\xi + 1)]$ . In particular  $\gamma_0$  is well-defined. Similarly,

$$e^2(e^\varsigma\xi + 1) = \omega^{e^\varsigma\xi \cdot \omega} = (\omega^{e^\varsigma\xi})^\omega = (e^\varsigma\xi)^\omega = \lim_{n \rightarrow \omega} (e^\varsigma\xi)^n.$$

For each  $n$ , there are  $(e^\varsigma\xi)^{n-1}$  points of nonzero  $\varsigma$ -rank in  $(e^\varsigma\xi)^n$  and so there are already  $e^2(e^\varsigma\xi + 1)$  points of nonzero  $\varsigma$ -rank in  $[1, e^2(e^\varsigma\xi + 1)]$ . The computation of  $\pi_1$  follows easily from this: a point  $\alpha$  is in  $X_\uparrow$  if, and only if,  $\ell^\varsigma\alpha \geq \xi$ . This can only happen if one of the following occur, from which an easy induction and the fact that  $e^2(e^\varsigma\xi + 1)$  is multiplicatively indecomposable yield the desired result:

1.  $\alpha_0 = \alpha_1 = 0$  and  $\ell^\varsigma\alpha_2 \geq \xi$  or  $\alpha_2$  is a successor ordinal, in which case

$$\alpha = \pi_1\alpha = e^2(e^\varsigma\xi + 1) \cdot \alpha_2.$$

2.  $\alpha_0 = 0$ ,  $\alpha_1 \neq 0$ , and  $\ell^\varsigma\alpha_1 \geq \xi$  or  $\alpha_1$  is a successor ordinal, in which case

$$\alpha = e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + e^1(e^\varsigma\xi + 1) \cdot \alpha_1$$

and

$$\pi_1\alpha = e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1.$$

3.  $\alpha_0 \neq 0$ , in which case  $\alpha_0$  is necessarily a successor ordinal and

$$\alpha = e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + e^1(e^\varsigma\xi + 1) \cdot \alpha_1 + (e^\varsigma\xi) \cdot \alpha_0,$$

and there are two possibilities: if  $\alpha_1$  is a limit ordinal whose  $\mathcal{I}_\varsigma$ -rank is less than  $\xi$ , then

$$e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + e^1(e^\varsigma\xi + 1) \cdot \alpha_1 \notin X_\uparrow,$$

and so

$$e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 = \pi_1\left(e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + e^1(e^\varsigma\xi + 1) \cdot \alpha_1 + (e^\varsigma\xi)\right);$$

otherwise,

$$e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + e^1(e^\varsigma\xi + 1) \cdot \alpha_1 \in X_\uparrow,$$

and so

$$e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 = \pi_1\left(e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + e^1(e^\varsigma\xi + 1) \cdot \alpha_1\right).$$

It follows from this that

$$\pi_1\alpha = \begin{cases} e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 + \gamma_0, & \text{if } \rho < \xi \text{ and } \alpha_1 \in \text{Lim} \\ e^2(e^\varsigma\xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 + \alpha_0, & \text{otherwise.} \end{cases}$$

holds in this case as well.

□

**Corollary 4.38.** *If  $\varsigma = 1$ , then  $\Theta = \kappa \cdot \omega \cdot \lambda$ , while if  $\varsigma$  is a limit ordinal, then*

$$\Theta = e^2(e^\varsigma \xi + 1) \cdot \lambda_2 + e^1(e^\varsigma \xi + 1) \cdot \lambda_1 + (e^\varsigma \xi) \cdot \lambda_0,$$

where  $\lambda_0, \lambda_1, \lambda_2$  are the unique ordinals such that:

1. One of the following hold:

(a)  $\lambda = e^2(e^\varsigma \xi + 1) \cdot \lambda_2 + \omega \cdot \lambda_1 + \lambda_0$ , or

(b)  $\lambda = e^2(e^\varsigma \xi + 1) \cdot \lambda_2 + \omega \cdot \lambda_1 + (\lambda_0 - 1)$ ,

and  $\lambda = e^2(e^\varsigma \xi + 1) \cdot \lambda_2 + \omega \cdot \lambda_1 + (\lambda_0 - 1)$  if, and only if,  $\lambda_1$  is a limit ordinal such that  $\ell^\varsigma \lambda_1 = 0$ .

2.  $\lambda_0 < \omega$ , and

3.  $\omega \cdot \lambda_1 + \lambda_0 < e^2(e^\varsigma \xi + 1)$ .

In particular, if  $\lambda > e^\varsigma \xi$  is multiplicatively indecomposable, then  $\Theta = \lambda$ .

*Proof.* Follows from Lemma 4.37 and the fact that  $e^2(e^\varsigma \xi + 1)$  is the least multiplicatively indecomposable ordinal greater than  $e^\varsigma \xi$ . □

**Corollary 4.39.** *Suppose  $1 < \mu$  and  $\alpha \in X_\uparrow$ . Write*

$$\alpha = e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + e^1(e^\varsigma \xi + 1) \cdot \alpha_1 + (e^\varsigma \xi) \cdot \alpha_0$$

and let  $i$  be least such that  $0 < \alpha_i$ . Then one of the following holds:

1.  $i = 0$ ,  $\ell^\mu \alpha = \ell^\mu e^\varsigma \xi$ , and  $\ell^\mu(\pi_1 \alpha) = 0$ .

2.  $i > 0$  and  $\ell^\mu \alpha = \ell^\mu(\pi_1 \alpha)$ .

*Proof.* This follows from (4.9) and Lemma 4.11.1. Note that if  $i = 1$ , then either  $\alpha_1$  is a limit ordinal, in which case 2 holds, or  $\alpha_1$  is a successor ordinal, in which case  $\ell^\mu \alpha = \ell^\mu(e^1(e^\varsigma \xi + 1)) = 0$ . Similarly,  $\ell^\mu \pi \alpha = \ell^\mu(\omega) = 0$ , so that 2 holds as well. □

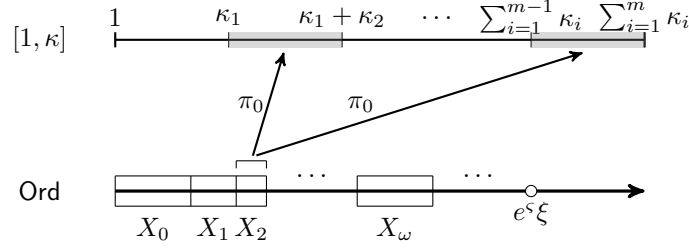
**Lemma 4.40.** *The projection function  $\pi_0$  has the following properties:*

1. If  $\iota \equiv i \pmod m$ , then  $\pi_0: (X_\iota, \mathcal{I}_\varsigma) \rightarrow ([1, \kappa_0] \sqcup [1, \kappa_i], \mathcal{I}_\varsigma)$  is a surjective  $d$ -map. If, in addition,  $\varsigma = 1$ , then it is a homeomorphism.

2.  $\pi_0: (X_\downarrow, \mathcal{I}_\varsigma) \rightarrow ([1, \kappa], \mathcal{I}_\varsigma)$  is a surjective  $d$ -map.

3.  $\pi_0(\beta_\iota) = \kappa_i$ , where  $\iota \equiv i \pmod m$ .

*Proof.* By Lemma 4.31, the sets  $Y_\iota$  and  $Z_\iota$  are  $\mathcal{I}_\varsigma$ -clopen in  $X_\iota$  and in each of those cases  $\pi_0$  defined as a combination of additions, subtractions, and logarithms and is thus a  $d$ -map. Moreover, if  $\varsigma = 1$ , then  $\mathbf{cs}(\varsigma)$  is the constant sequence with value 0 and so  $X_\iota$  is homeomorphic to  $[1, \kappa_0] \sqcup [1, \kappa_i]$ . This gives item 1. Here is the picture:



Item 2 is obtained by a similar argument, as each  $X_\iota$  is  $\mathcal{I}_\varsigma$ -clopen. Item 3 follows readily from the definition.  $\square$

**Lemma 4.41.**  $\pi_0^{-1}\alpha$  is  $\mathcal{I}_\varsigma$ -dense in  $X_\uparrow$  for any  $\alpha \in [1, \kappa]$ .

*Proof.* We can even provide witnesses for the density. Let  $i$  be least such that

1.  $\alpha \leq \kappa_1 + \dots + \kappa_i$ , and
2.  $\alpha_0 = -(\kappa_1 + \dots + \kappa_{i-1} + 1) + \alpha > 0$ .

If  $\varsigma = 1$ , then the result is clear, as any  $\mathcal{I}_1$ -neighborhood of any  $\beta \in X_\uparrow$  contains an interval of the form  $[\delta, \beta]_0$  and, by construction,  $\beta$  is a limit of endpoints of cells  $X_\iota$ . In particular, the interval  $[\delta, \beta]$  contains some cell  $X_\iota$  with  $\iota \equiv i \pmod m$  and  $\alpha \in \pi_0(X_\iota) = [1, \kappa_0] \sqcup [1, \kappa_i]$  by Lemma 4.40.1, from which the result follows.

So suppose  $\varsigma \neq 1$ . Let  $\beta \in X_\uparrow$ , so that  $\beta$  has  $\mathcal{I}_\varsigma$ -rank  $\rho \geq \xi$  and  $U$  be an  $\mathcal{I}_\varsigma$ -neighborhood of  $\beta$ . Factor  $\beta = e^\varsigma \xi \cdot \beta_0$ . We distinguish two cases:

**Case I:**  $\beta_0$  is a successor ordinal. Then  $\rho = \xi$  and we may assume without loss of generality that  $\beta_0 = 1$ . Since the  $\mathcal{I}_1$ -rank of  $\beta$  satisfies

$$\rho_{\mathcal{I}_1}\beta = e^\varsigma \rho > 0,$$

we can apply Lemma 4.15 (over the interval topology) to obtain a  $\mathcal{I}_\varsigma$ -neighborhood base of  $\beta$  consisting of sets of the form

$$(\eta, e^\varsigma \rho]_\gamma \cap [\delta, \beta]_0,$$

for  $\eta < e^\varsigma \rho$ ,  $\gamma < \varsigma$ , and  $\delta < \beta$ .

Hence, we may assume  $U$  is of the form  $(\eta, e^\varsigma \rho]_\gamma \cap [\delta, \beta]_0$ . We need to find some ordinal  $\chi \in U \cap X_\downarrow$  such that  $\pi_0 \chi = \alpha$ . Let  $\mu$  be some successor ordinal  $\equiv i \pmod m$  large enough so that

1.  $\gamma < \varsigma_\mu < \varsigma$ ,
2.  $\eta < e^{-\gamma + \varsigma_\mu}(1 + \kappa_0 + \alpha_0)$ , and
3.  $\delta < e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0)$ .

This is certainly possible, as it follows from Lemma 4.32 that:

$$\lim_{\iota \rightarrow \varsigma} e^{\varsigma_\iota}(1 + \kappa_0 + \alpha_0) = \lim_{\iota \rightarrow \varsigma} \alpha_\iota = e^\varsigma \xi.$$

**Claim 4.42.** Let  $\chi := e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0)$ . Then  $\chi \in U \cap X_\mu$ .

*Proof.* Clearly,  $\chi \in [\delta, \beta]_0$ . Now, on the one hand,

$$\eta < e^{-\gamma+\varsigma_\mu}(1 + \kappa_0 + \alpha_0) = \ell^\gamma(e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0)); \quad (4.10)$$

on the other,  $\varsigma_\mu < \varsigma$  and  $\ell^\varsigma \alpha < \xi$ , so that clearly we have

$$\ell^\gamma e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0) < \beta. \quad (4.11)$$

From (4.10) and (4.11) follows that  $e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0) \in U$ . It remains to prove that  $\chi \in X_\mu$ . Denote by  $\mu^*$  the immediate predecessor of  $\mu$ . Notice that

$$e^{\varsigma_{\mu^*}}(\kappa_0 + \kappa_{i-1}) < e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0) \leq e^{\varsigma_\mu}(\kappa_0 + \kappa_i), \quad (4.12)$$

so that  $e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0) \in X_\mu$ . This proves the claim.  $\square$

We show that  $\pi_0(e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0)) = \alpha$ . Since  $\iota \equiv \varsigma_\iota \pmod m$  for any  $\iota$  and

$$\ell^{\varsigma_\mu} e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0) > \kappa_0,$$

we have:

$$\begin{aligned} \pi_0(e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0)) &= \kappa_0 + \dots + \kappa_{i-1} + 1 + (-(1 + \kappa_0) + \ell^{\varsigma_\mu} e^{\varsigma_\mu}(1 + \kappa_0 + \alpha_0)) \\ &= \kappa_0 + \dots + \kappa_{i-1} + 1 + \alpha_0 \\ &= \kappa_0 + \dots + \kappa_{i-1} + 1 + (-(\kappa_0 + \dots + \kappa_{i-1} + 1) + \alpha) \\ &= \alpha. \end{aligned}$$

Since  $U$  was arbitrary, this finishes the proof in this case.

**Case II:**  $\beta_0$  is a limit ordinal. It is enough to consider the case  $\ell^\varsigma \beta_0 = \xi$ , as any  $\mathcal{I}_\varsigma$ -neighborhood of any point of higher rank contains a point of rank  $\xi$ . In this case, there are cofinal-many copies of  $(1, e^\varsigma \xi)_0$  in  $[\delta, \beta]_0$ . Focus on any such set, which will be of the form

$$(\eta, e^\varsigma \rho]_\gamma \cap [\delta^*, \delta^* + e^\varsigma \xi)_0 \subset U.$$

Treat this set as in Case I: define the corresponding ordinal  $\mu^*$  and use it to define  $\chi^* \in U \cap X_{\mu^*}$ , which will be of the form  $\delta^* + \chi$ , for some  $\chi < e^\varsigma \xi$  whose image under  $\pi_0$  is  $\alpha$ . Since  $\chi$  and  $\chi^*$  lie in isomorphic cells  $X_\mu$  and  $X_{\mu^*}$ , they have the same image under  $\pi_0$ .  $\square$

**Lemma 4.43.** *The projection function  $\pi_1$  has the following openness properties:*

1. *If  $\varsigma = 1$ , then  $\pi_1: (X_\uparrow, \mathcal{I}_\varsigma) \rightarrow ([1, \lambda], \mathcal{I}_\varsigma)$  is open.*
2.  *$\pi_1: (X_\uparrow, \mathcal{I}_{\varsigma+\iota}) \rightarrow ([1, \lambda], \mathcal{I}_{\varsigma+\iota})$  is open, for  $0 < \iota$ .*
3. *If  $\gamma$  is an  $\mathcal{I}_\varsigma$ -limit point in  $X_\uparrow$ , then it has a neighborhood base of sets whose images under  $\pi_1$  are open in  $\mathcal{I}_\varsigma$ .*

*Proof.* We prove 2. The case when  $\varsigma = 1$  can be shown easily, so we assume  $\varsigma$  is a limit ordinal. We show item 2. Note that  $X_\uparrow$  is open in  $\mathcal{I}_{\varsigma+\iota}$ . We show that  $\pi_1(\alpha, \beta]_\gamma$  is  $\mathcal{I}_{\varsigma+\iota}$ -open in  $\pi_1 X_\uparrow$  for every  $\gamma$ . There are essentially three cases. First, we use Corollary 4.39 to show that:

**Claim 4.44.**  $\pi_1((\alpha, \beta]_\gamma \cap X_\uparrow)$  is open whenever  $\gamma > 1$ .

*Proof.* Notice that  $x \in \pi_1((\alpha, \beta]_\gamma \cap X_\uparrow)$  if, and only if, one of the following holds:

1.  $x$  is of the form  $e^2(e^\varsigma \xi + 1) \cdot x_2 + \omega \cdot x_1$  or of the form  $e^2(e^\varsigma \xi + 1) \cdot x_2$ , and

$$\alpha < \ell^\gamma x \leq \beta.$$

2.  $x$  is of the form  $e^2(e^\varsigma \xi + 1) \cdot x_2 + \omega \cdot x_1 + x_0$ , with  $x_0$  nonzero, and  $\alpha \leq e^\varsigma \xi$ .

Hence, there are two possibilities:

1. If  $\alpha \leq e^\varsigma \xi < \beta$ , then

$$\pi_1((\alpha, \beta]_\gamma \cap X_\uparrow) = (\alpha, \beta]_\gamma \cup ([1, \lambda] \setminus \text{Lim}).$$

2. If  $e^\varsigma \xi < \alpha < \beta$ , then

$$\pi_1((\alpha, \beta]_\gamma \cap X_\uparrow) = (\alpha, \beta]_\gamma.$$

In any case,  $\pi_1((\alpha, \beta]_\gamma \cap X_\uparrow)$  is open.  $\square$

The case  $\gamma = 0$  is simpler. Here, we have

$$\pi_1((\alpha, \beta]_0 \cap X_\uparrow) = [\alpha', \beta')_0,$$

where  $\alpha'$  is least such that  $\alpha < \pi_1^{-1}\alpha' \leq \beta$  and  $\beta'$  is least such that  $\beta < \pi_1^{-1}\beta'$ . This set is  $\mathcal{I}_\varsigma$ -open unless  $\alpha'$  is an  $\mathcal{I}_\varsigma$ -limit in  $\pi_1 X_\uparrow$ . However, it follows from Lemma 4.37 that this can only happen if  $\alpha'$  is isolated in  $\mathcal{I}_{\varsigma+1}$  and so it poses no problem.

Finally, it remains to show:

**Claim 4.45.**  $\pi_1(\alpha, \beta]_1$  is an open  $\mathcal{I}_1$ -interval in  $\pi_1 X_\uparrow$ .

*Proof.* Let  $x \in [1, \lambda]$ . Note that  $x \in \pi_1(\alpha, \beta]_1$  if, and only if, one of the following hold:

1.  $x$  is of the form  $e^2(e^\varsigma \xi + 1) \cdot x_2$  and

$$\alpha < \ell(e^2(e^\varsigma \xi + 1) \cdot x_2) \leq \beta.$$

2.  $x$  is of the form  $e^2(e^\varsigma \xi + 1) \cdot x_2 + \omega \cdot x_1$  and

$$\alpha < \ell(e^1(e^\varsigma \xi + 1) \cdot x_1) \leq \beta.$$

3.  $x$  is of the form  $e^2(e^\varsigma \xi + 1) \cdot x_2 + \omega \cdot x_1 + x_0$ , where  $x_1 = e^\varsigma \xi \cdot \delta_1$ , and

$$\alpha < \ell(e^\varsigma \xi \cdot (x_0 + 1)) \leq \beta.$$

4.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + \omega \cdot x_1 + x_0$ , where  $x_1 = e^\varsigma\xi \cdot \delta_1 + \rho$  and  $0 < \rho$ , and

$$\alpha < \ell(e^\varsigma\xi \cdot x_0) \leq \beta.$$

The fact that  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2$  is equivalent to having

$$\ell x \geq e(e^\varsigma\xi + 1).$$

Hence, the first case holds if, and only if,

$$x \in [e(e^\varsigma\xi + 1), \infty)_1 \text{ and } x \in (\alpha, \beta]_1. \quad (4.13)$$

For the second case, notice that if  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + \omega \cdot x_1$ , then

$$\alpha < \ell(e^\varsigma\xi + 1) \cdot x_1 \leq \beta \text{ if, and only if, } \alpha < e^\varsigma\xi + 1 + \ell x_1 \leq \beta.$$

But in this case we have

$$\ell x = 1 + \ell x_1.$$

So it follows that:

$$\alpha < \ell(e^\varsigma\xi + 1) \cdot x_1 \leq \beta \text{ if, and only if, } \alpha < e^\varsigma\xi + \ell x \leq \beta.$$

This is equivalent to one of the following:

1.  $\alpha \leq e^\varsigma\xi$ ,  $\beta \geq e^\varsigma\xi$ , and  $\ell x \leq -e^\varsigma\xi + \beta$ .
2.  $\alpha > e^\varsigma\xi$ , and  $-e^\varsigma\xi + \alpha < \ell x \leq -e^\varsigma\xi + \beta$ .

Hence, the second case holds if, and only if,

$$x \in (0, e(e^\varsigma\xi + 1))_1 \text{ and } x \in (\alpha', -e^\varsigma\xi + \beta]_1, \quad (4.14)$$

where

$$\alpha' = \begin{cases} -1 & \text{if } \alpha \leq e^\varsigma\xi \\ -e^\varsigma\xi + \alpha & \text{otherwise.} \end{cases}$$

For the third and fourth cases, since  $x_0$  is necessarily finite, they both hold if, and only if,

$$x \in [0, 0]_1 \text{ and } \alpha < e^\varsigma\xi \leq \beta. \quad (4.15)$$

We use equations (4.13), (4.14), and (4.15) to prove the claim. There are several cases. In each of them,  $\pi_1(\alpha, \beta]_1$  is computed as shown below (note that one must intersect the resulting interval with  $[1, \lambda]_0$ ).

1. If  $\beta \leq e^\varsigma\xi$ , then  $\pi_1(\alpha, \beta]_1$  is empty.
2. If  $\alpha < e^\varsigma\xi < \beta < e(e^\varsigma\xi + 1)$ , then

$$\begin{aligned} \pi_1(\alpha, \beta]_1 &= [0, 0]_1 \cup \left( (0, e(e^\varsigma\xi + 1))_1 \cap (-1, -e^\varsigma\xi + \beta]_1 \right) \\ &= [0, 0]_1 \cup ((0, -e^\varsigma\xi + \beta]_1) \\ &= [0, -e^\varsigma\xi + \beta]_1 \end{aligned}$$

3. If  $\alpha < e^\varsigma \xi < e(e^\varsigma \xi + 1) \leq \beta$ , then

$$\begin{aligned}\pi_1(\alpha, \beta]_1 &= [0, 0]_1 \cup \left( (0, e(e^\varsigma \xi + 1))_1 \cap (-1, -e^\varsigma \xi + \beta]_1 \right) \\ &\quad \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= [0, 0]_1 \cup \left( (0, e(e^\varsigma \xi + 1))_1 \right) \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= [0, e(e^\varsigma \xi + 1)]_1 \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= [0, \beta]_1,\end{aligned}$$

since  $-e^\varsigma \xi + \beta = \beta$  in this case.

4. If  $\alpha = e^\varsigma \xi$  and  $\beta < e(e^\varsigma \xi + 1)$ , then

$$\begin{aligned}\pi_1(\alpha, \beta]_1 &= (0, e(e^\varsigma \xi + 1))_1 \cap (-1, -e^\varsigma \xi + \beta]_1 \\ &= (0, 1 + (-e^\varsigma \xi + \beta))_1.\end{aligned}$$

5. If  $\alpha = e^\varsigma \xi$  and  $e(e^\varsigma \xi + 1) \leq \beta$ , then

$$\begin{aligned}\pi_1(\alpha, \beta]_1 &= \left( (0, e(e^\varsigma \xi + 1))_1 \cap (-1, -e^\varsigma \xi + \beta]_1 \right) \\ &\quad \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= (0, e(e^\varsigma \xi + 1)]_1 \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= (0, \beta]_1,\end{aligned}$$

since  $-e^\varsigma \xi + \beta = \beta$  in this case.

6. If  $e^\varsigma \xi < \alpha < \beta < e(e^\varsigma \xi + 1)$ , then

$$\begin{aligned}\pi_1(\alpha, \beta]_1 &= (0, e(e^\varsigma \xi + 1))_1 \cap (-e^\varsigma \xi + \alpha, -e^\varsigma \xi + \beta]_1 \\ &= (-e^\varsigma \xi + \alpha, -e^\varsigma \xi + \beta]_1\end{aligned}$$

7. If  $e^\varsigma \xi < \alpha < e(e^\varsigma \xi + 1) \leq \beta$ , then

$$\begin{aligned}\pi_1(\alpha, \beta]_1 &= (-e^\varsigma \xi + \alpha, -e^\varsigma \xi + \beta]_1 \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= (-e^\varsigma \xi + \alpha, \beta]_1,\end{aligned}$$

since  $-e^\varsigma \xi + \beta = \beta$  in this case.

8. If  $e(e^\varsigma \xi + 1) \leq \alpha < \beta$ , then

$$\pi_1(\alpha, \beta]_1 = (\alpha, \beta]_1.$$

The claim follows.  $\square$

This proves item 2. The proof of item 3 is similar. Note that by the observation made at the end of the case  $\gamma = 0$ , it follows that  $\pi_1(\alpha, \beta]_0$  cannot be a singleton that is an  $\mathcal{I}_\varsigma$  limit point in  $\pi_1 X_\uparrow$  unless the preimage of that singleton was  $\mathcal{I}_\varsigma$ -isolated in  $X_\uparrow$ .  $\square$

**Lemma 4.46.** *The projection function  $\pi_1$  has the following continuity properties:*

1. *If  $U$  is  $\mathcal{I}_\varsigma$ -open, then  $\pi_1^{-1}U$  is open in  $X_\uparrow$  in the subspace topology.*
2. *Let  $0 < \mu$ . If  $U$  is  $\mathcal{I}_{\varsigma+\mu}$ -open, then so too is  $\pi_1^{-1}U$ .*

*Proof.* The proof of item 1 is just like the proof of Lemma 4.43.2. We will just consider the case when  $\varsigma$  is a limit ordinal and take care of open intervals in  $\mathcal{I}_\iota$ , for  $0 < \iota$ , as the other cases are proved easily. There are two claims:

**Claim 4.47.** *Let  $1 < \gamma$  and  $(\alpha, \beta]_\gamma$  be an interval. Then  $\pi_1^{-1}(\alpha, \beta]_\gamma$  is  $\mathcal{I}_\varsigma$ -open in  $X_\uparrow$ .*

*Proof.* The projection is not  $\mathcal{I}_\varsigma$ -rank preserving—there are points which are limit in  $[1, \lambda]$  but isolated in  $X_\uparrow$ . However, this will not be a problem. Notice that  $x \in \pi_1^{-1}(\alpha, \beta]_\gamma$  if, and only if, one of the following holds:

1.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1$  or of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2$ , and
$$\alpha < \ell^\gamma x \leq \beta.$$
2.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1 + (e^\varsigma\xi) \cdot x_0$ , with  $x_0$  nonzero, and  $\pi_1 x \in (\alpha, \gamma]_\gamma$ . If so, then necessarily either  $\pi_1 x$  is an isolated point in  $\pi_1 X_\uparrow$ , or  $x_0 = 1$  and  $x_1$  is a limit point such that  $\ell^\varsigma x_1 < \xi$ .

Hence,  $x \in \pi_1^{-1}(\alpha, \beta]_\gamma$  if, and only if,  $x \in (\alpha, \beta]_\gamma$ , unless  $\alpha = -1$ , in which case it might happen that  $x \in \pi_1^{-1}(\alpha, \beta]_\gamma$  and  $x \notin (\alpha, \beta]_\gamma$ . This would imply that  $x$  is of the form  $x' + e^\varsigma\xi$  and hence isolated in  $X_\uparrow$ . Therefore,

$$\pi_1^{-1}(\alpha, \beta]_\gamma = (\alpha, \beta]_\gamma \cup S,$$

where  $S$  is a discrete set. It follows that  $\pi_1^{-1}(\alpha, \beta]_\gamma$  is open.  $\square$

It remains to consider the case  $\gamma = 1$ .

**Claim 4.48.** *Let  $(\alpha, \beta]_1$  be an interval. Then  $\pi_1^{-1}(\alpha, \beta]_1$  is  $\mathcal{I}_\varsigma$ -open in  $X_\uparrow$ .*

*Proof.* Let  $x \in X_\uparrow$ . Note that  $\pi_1 x \in (\alpha, \beta]_1$  if, and only if, one of the following hold:

1.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2$  and
$$\alpha < \ell(e^2(e^\varsigma\xi + 1) \cdot x_2) \leq \beta.$$
2.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1$  and
$$\alpha < \ell(\omega \cdot x_1) \leq \beta.$$
3.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1 + e^\varsigma\xi \cdot x_0$ , where  $x_1 = e^\varsigma\xi \cdot \delta_1$  and  $x_0 > 1$ , and
$$\alpha < \ell(x_0 - 1) \leq \beta.$$
4.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1 + e^\varsigma\xi \cdot x_0$ , where  $x_1 = e^\varsigma\xi \cdot \delta_1 + \rho$  and  $0 < \rho$ , and
$$\alpha < \ell(x_0) \leq \beta.$$



5.  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1 + e^\varsigma\xi \cdot 1$ , where  $x_1 = e^\varsigma\xi \cdot \delta_1$ , and

$$\alpha < \ell(\omega \cdot x_1) \leq \beta.$$

As before, the fact that  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2$  is equivalent to having

$$\ell x \geq e(e^\varsigma\xi + 1).$$

Hence, the first case holds if, and only if,

$$x \in [e(e^\varsigma\xi + 1), \infty)_1 \text{ and } x \in (\alpha, \beta]_1. \quad (4.16)$$

For the second case, notice that if  $x$  is of the form  $e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1$ , then

$$\alpha < \ell(\omega \cdot x_1) \leq \beta \text{ if, and only if, } \alpha < 1 + \ell x_1 \leq \beta.$$

But in this case, we have

$$\ell x = e^\varsigma\xi + 1 + \ell x_1.$$

And so it follows that

$$\alpha < \ell(\omega \cdot x_1) \leq \beta \text{ if, and only if, } \alpha < 1 + (-e^\varsigma\xi + \ell x) \leq \beta.$$

This is equivalent to one of the following:

1.  $\alpha \leq 2$ ,  $\beta \geq 2$ , and  $\ell x \leq e^\varsigma\xi + (-1 + \beta)$ .
2.  $\alpha > 2$ , and  $e^\varsigma\xi + (-1 + \alpha) < \ell x \leq e^\varsigma\xi + (-1 + \beta)$ .

Hence, the second case holds if, and only if,

$$x \in (e^\varsigma\xi, e(e^\varsigma\xi + 1))_1 \text{ and } x \in (\alpha', e^\varsigma\xi + (-1 + \beta)]_1, \quad (4.17)$$

where

$$\alpha' = \begin{cases} -1 & \text{if } \alpha \leq 2 \\ e^\varsigma\xi + (-1 + \alpha) & \text{otherwise.} \end{cases}$$

For the third and fourth cases, since  $x_0$  is finite, they both hold if, and only if,

$$x \in [e^\varsigma\xi, e^\varsigma\xi]_1 \text{ and } \alpha = -1. \quad (4.18)$$

Ordinals in the fifth case will pose no problems, as they must necessarily be  $\mathcal{I}_\varsigma$ -isolated in  $X_\uparrow$ . Let

$$A_{\alpha, \beta} = \{x \in X_\uparrow : x = e^2(e^\varsigma\xi + 1) \cdot x_2 + e^1(e^\varsigma\xi + 1) \cdot x_1 + e^\varsigma\xi \cdot 1, \\ x_1 = e^\varsigma\xi \cdot \delta_1, x_1 \in (-1 + \alpha, -1 + \beta]_1\}.$$

Clearly  $A_{\alpha, \beta}$  is open and has rank zero in  $X_\uparrow$ . We use equations (4.16), (4.17), and (4.18) to prove the claim. There are several cases. In each of them,  $\pi_1^{-1}(\alpha, \beta]_1$  is computed as shown below, with the exception that one must intersect the resulting interval with  $X_\uparrow$ .

1. If  $-1 = \alpha < \beta < 2$ , then

$$\pi_1^{-1}(\alpha, \beta)_1 = [e^\varsigma \xi, e^\varsigma \xi]_1.$$

2. If  $-1 = \alpha < 2 < \beta < e(e^\varsigma \xi + 1)$ , then

$$\begin{aligned} \pi_1^{-1}(\alpha, \beta)_1 &= [e^\varsigma \xi, e^\varsigma \xi]_1 \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cap \left( -1, e^\varsigma \xi + (-1 + \beta) \right)_1 \right) \\ &= [e^\varsigma \xi, e^\varsigma \xi]_1 \cup \left( e^\varsigma \xi, e^\varsigma \xi + (-1 + \beta) \right)_1 \\ &= [e^\varsigma \xi, e^\varsigma \xi + (-1 + \beta)]_1. \end{aligned}$$

3. If  $-1 = \alpha < e(e^\varsigma \xi + 1) \leq \beta$ , then

$$\begin{aligned} \pi_1^{-1}(\alpha, \beta)_1 &= [e^\varsigma \xi, e^\varsigma \xi]_1 \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cap \left( -1, e^\varsigma \xi + (-1 + \beta) \right)_1 \right) \\ &\quad \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= [e^\varsigma \xi, e^\varsigma \xi]_1 \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \right) \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= [e^\varsigma \xi, e(e^\varsigma \xi + 1)]_1 \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= [e^\varsigma \xi, \beta]_1. \end{aligned}$$

4. If  $\alpha = 0, 1, 2$  and  $\beta < e(e^\varsigma \xi + 1)$ , then

$$\begin{aligned} \pi_1^{-1}(\alpha, \beta)_1 &= A_{\alpha, \beta} \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cap \left( -1, e^\varsigma \xi + (-1 + \beta) \right)_1 \right) \\ &= A_{\alpha, \beta} \cup \left( e^\varsigma \xi, e^\varsigma \xi + (-1 + \beta) \right)_1. \end{aligned}$$

5. If  $\alpha = 0, 1, 2$  and  $e(e^\varsigma \xi + 1) \leq \beta$ , then

$$\begin{aligned} \pi_1^{-1}(\alpha, \beta)_1 &= A_{\alpha, \beta} \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cap \left( -1, e^\varsigma \xi + (-1 + \beta) \right)_1 \right) \\ &\quad \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= A_{\alpha, \beta} \cup \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= A_{\alpha, \beta} \cup (e^\varsigma \xi, \beta]_1. \end{aligned}$$

6. If  $2 < \alpha < \beta < e(e^\varsigma \xi + 1)$ , then

$$\begin{aligned} \pi_1^{-1}(\alpha, \beta)_1 &= A_{\alpha, \beta} \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cap \left( e^\varsigma \xi + (-1 + \alpha), e^\varsigma \xi + (-1 + \beta) \right)_1 \right) \\ &= A_{\alpha, \beta} \cup (e^\varsigma \xi + (-1 + \alpha), e^\varsigma \xi + (-1 + \beta)]_1. \end{aligned}$$

7. If  $2 < \alpha < e(e^\varsigma \xi + 1) \leq \beta$ , then

$$\begin{aligned} \pi_1^{-1}(\alpha, \beta)_1 &= A_{\alpha, \beta} \cup \left( \left( e^\varsigma \xi, e(e^\varsigma \xi + 1) \right)_1 \cap \left( e^\varsigma \xi + (-1 + \alpha), e^\varsigma \xi + (-1 + \beta) \right)_1 \right) \\ &\quad \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= A_{\alpha, \beta} \cup \left( e^\varsigma \xi + (-1 + \alpha), e(e^\varsigma \xi + 1) \right)_1 \cup [e(e^\varsigma \xi + 1), \beta]_1 \\ &= A_{\alpha, \beta} \cup (e^\varsigma \xi + (-1 + \alpha), \beta]_1. \end{aligned}$$

8. If  $e(e^\varsigma \xi + 1) \leq \alpha < \beta$ , then

$$\pi_1^{-1}(\alpha, \beta]_1 = A_{\alpha, \beta} \cup (\alpha, \beta]_1.$$

This proves the claim.  $\square$

Item 1 is a consequence of the claim. Item 2 follows from  $1 \dashv X_\uparrow$  is open in  $\mathcal{I}_{\varsigma+\mu}$  whenever  $0 < \mu$ . This proves the lemma.  $\square$

Now that we have proved important facts about the projection functions, we are ready for the final step. We will consider minimal algebras and maximal algebras separately.

#### 4.4.3 Minimal algebras

In this case, the polytopology  $\{\mathcal{R}_\iota\}_{\iota < \Lambda}$  we will consider will be Icard. What remains is to define the appropriate algebra  $\mathcal{C}$ . Fix two minimal GLP-ambiances  $([1, \kappa], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  and  $([1, \lambda], \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  over  $\mathcal{I}_\varsigma$ . We define  $\mathcal{C}$  to be the Boolean algebra given by the sets:

$$\pi_0^{-1}A \cup \pi_1^{-1}B, \text{ for } A \in \mathcal{A}, B \in \mathcal{B},$$

and by the set:

$$S = (X_\uparrow \setminus dX_\uparrow) \cap (\pi_1^{-1}d[1, \lambda]),$$

where  $d = d_{\mathcal{I}_\varsigma}$ . Thus,  $S$  is the set of points  $x$  such that:

1.  $x$  is isolated in  $X_\uparrow$ , and
2.  $\pi_1 x$  is not isolated in  $\pi_1 X_\uparrow$ .

As we show, adding the set  $S$  to the algebra will not make it any more difficult to prove that it is a GLP-algebra.

**Lemma 4.49.** *Let  $d = d_{\mathcal{I}_\varsigma}$ . Then  $dS = \emptyset$ .*

*Proof.* Since  $S \subset X_\uparrow$ , any limit point of  $S$  must be a limit point of  $X_\uparrow$  and hence have  $\mathcal{I}_\varsigma$  rank  $\geq \xi + 1$ . However, every point in  $S$  is of the form:

$$x = e^2(e^\varsigma \xi + 1) \cdot x_2 + e^1(e^\varsigma \xi + 1) \cdot x_1 + e^\varsigma \xi \cdot 1,$$

where  $x_1$  is a limit ordinal satisfying  $\ell^\varsigma x_1 < \xi$ . But then if  $\gamma \in dX_\uparrow$ , then  $\ell\gamma \geq e^\varsigma(\xi + 1)$ , so that

$$U = (e^\varsigma \xi, \ell\gamma]_1$$

is a neighborhood of  $\gamma$  avoiding  $S$ .  $\square$

We need to show the following:

**Lemma 4.50.**  *$([1, \Theta], \{\mathcal{R}_\iota\}_{\iota < \Lambda}, \mathcal{C})$  is a minimal GLP-ambiance.*

*Proof.* We show that  $\mathcal{C}$  is closed under  $d_\iota = d_{\mathcal{I}_{\varsigma+\iota}}$ , for  $\iota < \Lambda$ , and that it satisfies (2.1). Let  $C = \pi_0^{-1}A \cup \pi_1^{-1}B \in \mathcal{C}$  and  $\iota < \Lambda$ . We show  $d_\iota C \in \mathcal{C} \cap \mathcal{I}_{\varsigma+\iota+1}$ . Because of the fact that, in general,

$$d_\iota(A \cup B) = d_\iota A \cup d_\iota B,$$

it suffices to check the following three cases:

**Case I.**  $A \neq \emptyset$  and  $0 = \iota$ , so that  $d_\iota = d_{\mathcal{I}_\varsigma}$ . By Lemma 4.41,  $X_\uparrow \in d_\iota A$ . The set  $X_\downarrow$  is  $\mathcal{I}_\varsigma$ -open and  $\pi_0$  is a d-map by Lemma 4.40.2, whence  $\alpha \in d_\iota C \cap X_\downarrow$  if, and only if,  $\pi_0 \alpha \in d_\iota A$ . Hence,

$$d_\iota C = \pi_0^{-1}d_\iota A \cup X_\uparrow \in \mathcal{C}.$$

Moreover,  $X_\uparrow \in \mathcal{I}_{\varsigma+1}$  and, by Lemma 4.2.3,

$$\pi_0: (X_\downarrow, \mathcal{I}_{\varsigma+1}) \rightarrow ([1, \kappa], \mathcal{I}_{\varsigma+1})$$

is a d-map, so that if  $d_\iota A$  is  $\mathcal{I}_{\varsigma+1}$ -open, then so too is  $d_\iota C$ .

**Case II.**  $A = \emptyset$  and  $0 = \iota$ , so that  $d_\iota = d_{\mathcal{I}_\varsigma}$  again. Then  $C = \pi_1^{-1}B$ . Suppose  $\alpha \in d_\iota C$ . Clearly  $\alpha \in X_\uparrow$ . First, consider the case  $\varsigma = 1$ , so that  $\alpha = \kappa \cdot \alpha_0$  and  $\pi_1 \alpha = \alpha_0$  by Lemma 4.37.1. We show that

$$d_\iota C = \pi_1^{-1}d_\iota B \tag{4.19}$$

This is indeed sufficient, for since  $([1, \lambda], \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  is a  $\text{GLP}_\Lambda$ -ambiance, then  $d_\iota B \in \mathcal{I}_{\varsigma+1}$ , whence  $\pi_1^{-1}d_\iota B \in \mathcal{I}_{\varsigma+1}$  by Lemma 4.46.2.

Let  $U$  be an  $\mathcal{I}_1$ -neighborhood of  $\alpha_0$  in  $[1, \lambda]$ . Then  $\pi_1^{-1}U$  is of the form  $(\gamma, \delta] \cap X_\uparrow$ . Since  $\alpha \in d_\iota C$ , then for every  $\mathcal{I}_\varsigma$ -neighborhood  $V$  of  $\alpha$ , there is some  $\beta \in V \cap C$ . If we consider  $V = (\gamma, \delta]$ , we must have  $\beta \in \pi_1^{-1}B$ , as it cannot be in  $X_\downarrow$  because  $C \cap X_\downarrow = \emptyset$ . Hence,  $\pi_1 \beta \in U$  and  $\alpha_0 \in d_\iota B$ , i.e.,  $\alpha \in \pi_1^{-1}d_\iota B$ .

Conversely, suppose  $\pi_1 \alpha \in d_\iota B$ . Let  $U$  be a  $\mathcal{I}_\varsigma$ -neighborhood of  $\alpha$ . Then  $\pi_1 U \in \mathcal{I}_\varsigma$  by Lemma 4.43.1, whence it intersects  $B$  at some ordinal  $\beta$ . Hence,  $\pi_1^{-1}\beta \in \pi_1^{-1}B = C$  and so  $\alpha \in d_\iota C$ . This proves (4.19).

Now consider the case that  $\varsigma$  is a limit ordinal, so that  $\alpha$  and  $\pi_1 \alpha$  are given by (4.8) and (4.9), respectively. We claim that

$$d_\iota C = \pi_1^{-1}d_\iota B \setminus S. \tag{4.20}$$

This is sufficient, as—given (4.20)— $\pi_1^{-1}d_\iota B$  is  $\mathcal{I}_{\varsigma+1}$ -open in  $X_\uparrow$ , so that, if  $x \in d_\iota C$ , there is a natural neighborhood  $U$  of  $x$  given by  $\pi_1$ , whence one obtains a neighborhood of  $x$  contained in  $d_\iota C$  by intersecting  $U$  with  $(e^\varsigma \xi, \infty)_1$ .

To show that  $d_\iota C \subset \pi_1^{-1}d_\iota B \setminus S$ , the same argument as above goes through: let  $\alpha \in d_\iota C$  and  $U$  be an  $\mathcal{I}_\varsigma$ -neighborhood of  $\pi_1 \alpha$ . The following hold:

1.  $\pi_1^{-1}U$  is open in the subspace topology by Lemma 4.46.1;
2.  $\alpha \in d_\iota C$ ;
3.  $C$  intersects any open set containing  $\pi_1^{-1}U$  at a point of  $\mathcal{I}_\varsigma$ -rank  $\xi$ .

It follows that  $C$  intersects  $\pi_1^{-1}U$  at a point  $\beta$ . Hence,  $\pi_1\beta \in U$  and so  $\pi_1\alpha \in d_\iota B$ , whence  $\alpha \in \pi_1^{-1}d_\iota B$ . Clearly,  $\alpha \notin S$ .

Conversely, suppose  $\pi_1\alpha \in d_\iota B$  and  $\alpha \notin S$ . We show that  $\alpha \in d_\iota C$ . By definition,  $X_\uparrow \setminus S$  is the set of points  $x$  in  $X_\uparrow$  such that one of the following holds:

1.  $x$  is a limit point in  $X_\uparrow$ , or
2.  $\pi_1 x$  is isolated in  $\pi_1 X_\uparrow$ .

It follows that  $\alpha$  is a limit point in  $X_\uparrow$ . Let  $U$  be an  $\mathcal{I}_\zeta$ -neighborhood of  $\alpha$ . Then we might assume that  $\pi_1 U \in \mathcal{I}_\zeta$  by Lemma 4.43.3, whence it intersects  $B$  at some ordinal  $\beta$ . Hence,  $\pi_1^{-1}\beta \in \pi_1^{-1}B = C$  and so  $\alpha \in d_\iota C$ . This proves (4.20).

**Case III.**  $0 < \iota$ . Then  $\pi_0$  is a d-map by Lemmata 4.40.2 and 4.2.3 and  $\pi_1$  is a d-map by Lemmata 4.43.2 and 4.46.1 and the fact that it is a bijection. Since  $X_\uparrow$  is  $\mathcal{I}_{\zeta+\iota}$ -open, then

$$\begin{aligned} d_\iota C &= d_\iota(\pi_0^{-1}A \cup \pi_1^{-1}B) \\ &= d_\iota\pi_0^{-1}A \cup d_\iota\pi_1^{-1}B \\ &= \pi_0^{-1}d_\iota A \cup \pi_1^{-1}d_\iota B \in \mathcal{C}. \end{aligned}$$

Moreover, since  $([1, \kappa], \{\mathcal{T}_\iota\}_{\iota < \Lambda}, \mathcal{A})$  and  $([1, \lambda], \{\mathcal{S}_\iota\}_{\iota < \Lambda}, \mathcal{B})$  over  $\mathcal{I}_\zeta$  are GLP $_\Lambda$ -ambiances, it follows that  $d_\iota C \in \mathcal{I}_{\zeta+\iota+1}$ .  $\square$

#### 4.4.4 Maximal Algebras

In this case, we need to define a BG-space on  $([1, \Theta], \mathcal{I}_\zeta)$ .

**Lemma 4.51.** *There exists a BG-space  $(X_\downarrow, \{\hat{\mathcal{T}}_\iota\}_{\iota < \Lambda})$  over  $\mathcal{I}_\zeta$  such that  $\pi_0 : (X_\downarrow, \hat{\mathcal{T}}_\iota) \rightarrow ([1, \kappa], \mathcal{T}_\iota)$  is a d-map for each  $\iota$ .*

*Proof.* Notice that  $\pi_0 : (X_\downarrow, \mathcal{I}_\zeta) \rightarrow ([1, \kappa], \mathcal{I}_\zeta)$  is a d-map by Lemma 4.40.2. We build the required polytopology using Lemma 4.16.  $\square$

The polytopology  $\{\hat{\mathcal{T}}_\iota\}_{\iota < \Lambda}$  constructed in Lemma 4.51 is, however, not a topology on  $[1, \Theta]$ . We will, nonetheless, make it into a topology on  $[1, \Theta]$  by considering its set-theoretic union

$$\hat{\mathcal{T}}_\iota^* := \mathcal{I}_\zeta \cup \hat{\mathcal{T}}_\iota.$$

In the following series of lemmata, we will make use of various additional auxiliary topologies, which we list here for convenience:

**Definition 4.52.** As stated before, we set  $\hat{\mathcal{T}}_\iota^* := \mathcal{I}_\zeta \cup \hat{\mathcal{T}}_\iota$ . Moreover, we define:

- $\hat{\mathcal{S}}_\iota = \{\pi_1^{-1}U : U \in \mathcal{S}_\iota\} \subset \wp(X_\uparrow)$ , for  $0 < \iota$ ;
- $\mathcal{R}_\iota = \hat{\mathcal{T}}_\iota \sqcup \hat{\mathcal{S}}_\iota \subset \wp([1, \Theta])$ , for  $0 < \iota$ ;
- $\mathfrak{T}$  denotes the collection of topologies  $\delta$  that are limit-extensions of  $\hat{\mathcal{T}}_0^*$  and such that  $\mathcal{R}_1$  is a limit-extension of  $\delta_{\uparrow 1} = \mathcal{I}_{\zeta+1} \cup \delta$ .

We left the topology  $\mathcal{R}_0$  undefined—in fact, we will never give an explicit definition of  $\mathcal{R}_0$ . The last piece of the proof of Lemma 4.21—that a suitable  $\mathcal{R}_0$  exists and that it forms a BG-space over  $\mathcal{I}_\varsigma$  together with the other  $\mathcal{R}_\iota$ ’s—is the statement of Lemma 4.60 below.

**Lemma 4.53.**

1. The spaces  $([1, \lambda], \mathcal{S}_\iota)$  and  $(X_\uparrow, \hat{\mathcal{S}}_\iota)$  are homeomorphic, for  $0 < \iota$ .
2. The rank function on  $\hat{\mathcal{T}}_0^*$  is  $\ell^\varsigma$ .
3. The rank function on  $\mathcal{R}_1$  is  $\ell^{\varsigma+1}$ .

*Proof.* 1 is clear by definition of  $\hat{\mathcal{S}}_\iota$  and fact that  $\pi_1$  is a bijection. Item 2 follows from the fact that  $X_\downarrow$  is  $\mathcal{I}_\varsigma$ -open in  $[1, \Theta]$ , Lemma 4.51, as it implies that  $\hat{\mathcal{T}}_0$  is a rank-preserving extension of  $\mathcal{I}_\varsigma$ .

We prove item 3. Assume  $\alpha \in [1, \Theta]$ . If  $\alpha \in X_\downarrow$ , then

$$\rho_{\mathcal{R}_1} \alpha = \rho_{\hat{\mathcal{T}}_1} \alpha = \ell^{\varsigma+1} \alpha,$$

as by Lemma 4.51,  $\hat{\mathcal{T}}_1$  is a rank-preserving extension of  $(\hat{\mathcal{T}}_0)_{\uparrow 1}$  and  $\hat{\mathcal{T}}_0$  is a rank-preserving extension of  $\mathcal{I}_\varsigma$ .

If  $\alpha \in X_\uparrow$ , then

$$\rho_{\mathcal{R}_1} \alpha = \rho_{\hat{\mathcal{S}}_1} \alpha = \rho_{\mathcal{S}_1} \pi_1 \alpha = \rho_{\mathcal{S}_1} (1 + \mathbf{otyp}_{X_\uparrow} \alpha).$$

The second equality follows from item 1. By a similar argument as above, one shows that  $\rho_{\mathcal{S}_1} = \ell^{\varsigma+1}$ . If  $\varsigma = 1$ , then by Lemma 4.37.1,  $\alpha$  is of the form  $\kappa \cdot \omega \cdot \beta$ , and  $\mathbf{otyp}_{X_\uparrow} \alpha = \beta$ , whence  $\rho_{\mathcal{R}_1} \alpha = \ell^{\varsigma+1} (1 + \beta)$ . But

$$\ell^{\varsigma+1} (1 + \beta) = \ell^{\varsigma+1} (\kappa \cdot \omega \cdot \beta)$$

by Lemma 4.11.1 and Corollary 4.39. If  $\varsigma$  is a limit ordinal, then by Lemma 4.37.2,  $\alpha$  is of the form

$$e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + e^1(e^\varsigma \xi + 1) \cdot \alpha_1 + (e^\varsigma \xi) \cdot \alpha_0$$

and

$$\pi_1 \alpha = \begin{cases} e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 + \gamma_0, & \text{if } \alpha \in \text{Lim} \cap [0, \xi)_\varsigma \\ e^2(e^\varsigma \xi + 1) \cdot \alpha_2 + \omega \cdot \alpha_1 + \alpha_0, & \text{otherwise.} \end{cases}$$

The key point is that  $\xi$  is a successor ordinal. Hence, it follows from Corollary 4.39 that

$$\rho_{\mathcal{R}_1} \alpha = \rho_{\mathcal{S}_1} (1 + \mathbf{otyp}_{X_\uparrow} \alpha) = \ell^{\varsigma+1} \alpha_k = \ell^{\varsigma+1} \alpha,$$

where  $k$  is least such that  $\alpha_k$  is nonzero. □

**Lemma 4.54.** The polytopological space  $([1, \Theta], \{\mathcal{R}_{1+\iota}\}_{\iota < \Lambda})$  is a BG-space over  $([1, \Theta], \mathcal{R}_1)$ .

*Proof.* By Lemma 4.51,  $(X_\downarrow, \{\hat{\mathcal{T}}_\iota\}_{\iota < \Lambda})$  is a BG-space.  $(X_\uparrow, \{\hat{\mathcal{S}}_{1+\iota}\}_{\iota < \Lambda})$  is also a BG-space by Lemma 4.53.1. Since  $\mathcal{R}_1$  is an extension of  $\mathcal{I}_{\varsigma+1}$  on  $[1, \Theta]$ , it follows that  $X_\uparrow$  is  $\mathcal{I}_{\varsigma+1}$ -clopen in  $([1, \Theta], \mathcal{R}_1)$ . Hence, their topological sum is also BG-space over  $([1, \Theta], \mathcal{R}_1)$ . □

**Lemma 4.55.** *The family  $\mathfrak{T}$  has the following properties:*

1.  $\mathfrak{T}$  is nonempty.
2. Suppose  $\theta$  is maximal in  $\mathfrak{T}$  (with respect to inclusion). Then  $\theta$  is limit-maximal.
3. If  $\{\delta_\gamma\}_{\gamma < \eta}$  is a totally ordered subset of  $\mathfrak{T}$ , then  $\bigcup_{\gamma < \eta} \delta_\gamma \in \mathfrak{T}$ .

*Proof.* Item 1. For a point  $y \in \mathcal{S}_0$ , denote by  $\mathbf{nhd}_y(\mathcal{S}_0)$  the set of all neighborhoods of  $y$ . Let  $\mathcal{U}$  be the family of  $\mathcal{S}_0$ -open sets defined by

$$\mathcal{U} = \{U \in \mathbf{nhd}_y(\mathcal{S}_0) \setminus \mathcal{I}_\varsigma : 0 < \ell^{\varsigma+1}y, \{y\} = \ell^\varsigma(U) \cap [\ell^\varsigma y, \infty)_\varsigma\},$$

i.e.,  $\mathcal{U}$  is the family of nontrivial  $\mathcal{S}_0$ -open sets not in  $\mathcal{I}_\varsigma$  that contain a unique point of maximal rank. For any ordinal  $\alpha$  such that  $0 = \ell^{\varsigma+1}\alpha$ , write

$$\alpha = e^{\varsigma+1}1 \cdot \alpha_0 + \alpha_1,$$

where  $\alpha_1$  is minimal nonzero. For each ordinal  $\alpha$ , set:

$$O_\alpha = \begin{cases} (e^{\varsigma+1}1 \cdot \alpha_0, e^{\varsigma+1}1 \cdot \alpha_0 + \alpha_1] & \text{if } 0 = \ell^{\varsigma+1}\alpha \\ \emptyset & \text{if } 0 < \ell^{\varsigma+1}\alpha; \end{cases}$$

For each  $U \in \mathcal{U}$ , the set  $U_O \subset [1, \Theta]$  is defined by

$$U_O = \pi_1^{-1}U \cup \bigcup_{\alpha \in U} O_{\pi_1^{-1}\alpha}.$$

Define  $\mathcal{T}_U$  to be the topology generated by  $\hat{\mathcal{T}}_0^*$  and the family  $\{U_O : U \in \mathcal{U}\}$ . It suffices to show:

**Claim 4.56.**  $\mathcal{T}_U$  is in  $\mathfrak{T}$ .

*Proof.* We first need to show that  $\mathcal{T}_U$  is a limit-extension of  $\hat{\mathcal{T}}_0^*$ .

**Subclaim 4.57.** *Suppose  $\alpha \in [1, \lambda]$ . Then  $\ell^\varsigma(O_{\pi_1^{-1}\alpha})$  is an initial segment of Ord. If, in addition,  $0 = \ell^{\varsigma+1}\alpha$  and  $\varsigma = 1$ , then it extends  $[0, L\kappa]$ .*

*Proof.* If  $0 < \ell^{\varsigma+1}\alpha$ , we have that

1.  $\xi$  is a successor ordinal,
2.  $0 < \ell^{\varsigma+1}\pi_1^{-1}\alpha$ ,
3. the set  $\ell^\varsigma(O_{\pi_1^{-1}\alpha})$  is empty,

where the second item follows from the first and the third follows from the second, together with Corollary 4.39 and the fact that  $\varsigma \geq 1$ . Suppose instead that  $0 = \ell^{\varsigma+1}\alpha$ . Again, Corollary 4.39 gives  $0 = \ell^{\varsigma+1}\pi_1^{-1}\alpha$ , so that for some  $\alpha_0$  and some minimal  $\alpha_1 < e^{\varsigma+1}1$ ,

$$\begin{aligned} \ell^\varsigma(O_{\pi_1^{-1}\alpha}) &= \ell^\varsigma(e^{\varsigma+1}1 \cdot \alpha_0, e^{\varsigma+1}1 \cdot \alpha_0 + \alpha_1] \\ &= \ell^\varsigma[0, \alpha_1]. \end{aligned}$$

The latter is clearly an initial segment. Moreover, if  $\varsigma = 1$ , then

$$\ell^\varsigma[0, \alpha_1] = L\alpha_1 + 1$$

by Lemma 3.9. Since  $\pi_1^{-1}\alpha \in X_\uparrow$ , then

$$L\alpha_1 \geq \ell\alpha_1 = \ell\alpha \geq \xi.$$

This yields the subclaim, since

$$\xi = \ell^\varsigma[1, \kappa] = L\kappa + 1.$$

□

**Subclaim 4.58.**  $\mathcal{T}_U$  is a rank-preserving extension of  $\hat{\mathcal{T}}_0^*$ .

*Proof.* By Lemmata 2.3 and 4.53.2 and the definition of  $\mathcal{T}_U$ , it suffices to show that  $\ell^\varsigma(U_O)$  is an initial segment of  $\mathbf{Ord}$  for each  $U \in \mathcal{U}$ . Since

$$U_O = \pi_1^{-1}U \cup \bigcup_{\alpha \in U} O_{\pi_1^{-1}\alpha},$$

we have that

$$\ell^\varsigma(U_O) = \ell^\varsigma(\pi_1^{-1}U) \cup \bigcup_{\alpha \in U} \ell^\varsigma(O_{\pi_1^{-1}\alpha})$$

By the previous subclaim,  $\ell^\varsigma(O_{\pi_1^{-1}\alpha})$  is an initial segment of  $\mathbf{Ord}$  for any  $\alpha$  and if  $0 = \ell^{\varsigma+1}\alpha$  and  $\varsigma = 1$ , then  $\ell^\varsigma(O_{\pi_1^{-1}\alpha})$  extends  $[0, L\kappa]$ . Moreover, if  $\varsigma = 1$ , then

$$\ell^\varsigma(\pi_1^{-1}U) = [L\kappa, L\kappa + \ell^\varsigma(U)],$$

whence  $\ell^\varsigma(U_O)$  is an initial segment. On the other hand, if  $\varsigma$  is a limit ordinal, then  $\ell^\varsigma(\pi_1^{-1}U) = \ell^\varsigma(U)$  is also an initial segment. This proves the subclaim. □

We show  $\mathcal{T}_U$  is a limit-extension of  $\hat{\mathcal{T}}_0^*$ . Suppose  $x \in U_O$  is of successor rank, i.e.,  $\ell^{\varsigma+1}x = 0$ . If  $x \in X_\downarrow$ , then  $x \in O_\alpha \subset U_O$ , for some  $\alpha$ . Since  $O_\alpha \in \mathcal{I}_1$ , we are done. If, on the other hand,  $x \in X_\uparrow$ , then it follows that  $\ell^{\varsigma+1}(\pi_1 x) = 0$  and so the set  $O_x$  is a subset of  $U_O$ .

Finally,  $\mathcal{R}_1$  has the same rank function as  $(\mathcal{T}_U)_{\uparrow 1}$  by Lemma 4.55.3. That  $\mathcal{R}_1$  is a limit-extension of  $\mathcal{T}_{U\uparrow 1}$  follows from the facts that

$$(\mathcal{T}_U)_{\uparrow 1} = \mathcal{I}_{\varsigma+1} \cup \mathcal{T}_U$$

(this is because  $\mathcal{T}_U$  is a rank-preserving extension of  $\mathcal{I}_\varsigma$ ) and that for each  $U_O \in \mathcal{T}_U \setminus \mathcal{I}_\varsigma$ , the set  $\pi_1(U_O)$  is in  $\mathcal{S}_0$ , whence

$$\pi_1^{-1}\pi_1(U_O) \in \hat{\mathcal{S}}_1 \subset \mathcal{R}_1.$$

This finishes the proof of the claim, and, hence, of item 1. □

To show item 2, we use Lemma 3.3. Limit-maximality of  $\theta$  in  $X_\downarrow$  follows from the fact that  $\theta$  extends  $\hat{\mathcal{T}}_0$ . Hence, let  $x \in X_\uparrow$  be such that  $0 < \ell^{\varsigma+1}x = \ell\rho_\theta x$  and  $V$  be  $\theta$ -open and consisting of points of  $\theta$ -rank below  $x$ . Assume, towards a contradiction, that  $V \cup \{x\} \notin \theta$ . Let  $\theta_*$  be generated by  $\theta$  and  $V \cup \{x\}$ , so  $\theta_*$



only differs from  $\theta$  around  $x$ . Since  $0 < \ell^{\varsigma+1}x$  and  $\ell^{\varsigma}(V \cup \{x\}) = [0, \ell^{\varsigma}x]$  is an initial segment,  $\theta_*$  is a limit-extension of  $\theta$  by Lemma 2.3. Moreover, the rank function on  $\theta_{*\uparrow 1}$  is  $\ell^{\varsigma+2}$  and so  $\mathcal{R}_1$  is clearly a limit-extension of  $\theta_{*\uparrow 1}$ , contradicting the  $\mathfrak{T}$ -maximality of  $\theta$ .

We prove item 3. Let  $\delta = \bigsqcup_{\gamma < \eta} \delta_\gamma$ . The family  $\{\delta_\gamma\}_\gamma$  is non-decreasing and has a common rank function, whereby so too does  $\delta$ . Moreover, no topology  $\delta_\gamma$  adds any neighborhood around points of successor rank, and so neither does  $\delta$ . Hence, it is a limit-extension of  $\hat{\mathcal{T}}_0^*$ . Similarly, the fact that  $\mathcal{R}_1$  is a limit-maximal extension of any one  $(\delta_\gamma)_{\uparrow 1}$  implies that it is so as well of  $\delta_{\uparrow 1}$ .  $\square$

Zorn's Lemma and Lemma 4.55 together yield:

**Corollary 4.59.** *There is a limit-maximal extension  $\mathcal{R}_0$  of  $\hat{\mathcal{T}}_0^*$  on  $[1, \Theta]$  such that  $\mathcal{R}_1$  is a limit-maximal extension of  $(\mathcal{R}_0)_{\uparrow 1}$ .*

**Lemma 4.60.**  *$\{\mathcal{R}_\iota\}_{\iota < \Lambda}$  is a BG-polytopology over  $([1, \Theta], \mathcal{I}_\varsigma)$ .*

*Proof.* By Lemma 4.53.2 and Lemma 4.54.  $\square$

This completes the proof of Lemma 4.21.  $\square$

## 5 Finite-support completeness

It seems desirable to attain completeness with respect to polytopologies based on the simplest example of scattered spaces for which it is not unreasonable to expect it—the interval topology. It is not clear at all whether this is possible; all existent machinery breaks down with the introduction of uncountably many modalities. In spite of this, in this section we prove a result related to Theorem 4.20 that is meant to be taken as evidence towards a positive answer.

In this vein, let  $\tau^\kappa$  denote *any* topology obtained as follows:  $\tau^\kappa$  is the  $\kappa$ th topology in a BG-space over some rank-preserving extension of the interval topology. It follows from the results in this section that if, for each suitably closed<sup>5</sup>  $\Theta$  and  $\kappa$ , there exists a d-map

$$f: (\Theta, \tau^\kappa) \rightarrow (\Theta, \mathcal{I}_\kappa), \quad (5.1)$$

then  $\text{GLP}_\Lambda$  is complete with respect to some BG-space over a rank-preserving extension of the interval topology. It is not clear whether this is possible. However, it follows from the pullback lemma that logarithms are d-maps

$$\ell^\xi: (\Theta, \tau_{\uparrow(-\xi+\kappa)}^\xi) \rightarrow (\Theta, \mathcal{I}_\kappa),$$

for additively indecomposable  $\kappa$ . We use this observation to prove an analog of the completeness theorem for BG-polytopologies for their 'finite-support' counterparts. These allow us to manage our resources more efficiently and require a lower assumption on the height of the original space.

**Definition 5.1** ( $\vartheta$ -maximal topology). Let  $\vartheta$  be a nonzero ordinal and  $(X, \tau)$  be a scattered topological space. We say that  $\tau_*$  is an  $\vartheta$ -extension of  $\tau$  if

<sup>5</sup>For example, it would suffice to assume  $\kappa$  is additively indecomposable and  $\Theta \geq e^\kappa 1$ .

1.  $\tau \subset \tau_*$ ,
2.  $\rho_{\tau_*} = \rho_\tau$ , and
3. the identity function  $id : (X, \tau) \rightarrow (X, \tau_*)$  is continuous at all points  $x$  such that

$$\ell^\vartheta \rho_\tau(x) = 0.$$

We say that  $\tau_*$  is an  $\vartheta$ -maximal topology if there are no proper  $\vartheta$ -extensions of  $\tau_*$ .

In particular, when  $\vartheta = 1$ , the notions of  $\vartheta$ -maximality and limit-maximality coincide. Any  $\vartheta_1$ -maximal topology is also  $\vartheta_2$ -maximal, provided  $\vartheta_1 < \vartheta_2$ . If  $\vec{\vartheta} = \{\vartheta_i : i < n\}$  is a finite increasing sequence of ordinals, we write

$$\partial \vec{\vartheta} := \{\partial \vartheta_{i+1} : i < n\},$$

where  $\partial \vartheta_{i+1} = -\vartheta_i + \vartheta_{i+1}$ .

**Definition 5.2.** Let us call a polytopological space  $\mathfrak{X} = (X, \tau_0, \dots, \tau_n)$  a  $\vec{\vartheta}$ -polytopology over  $(X, \tau)$  if  $\vec{\vartheta} = \{\vartheta_i : i \leq n\}$  is increasing and

1.  $\tau_0$  is a  $\vartheta_0$ -maximal extension of  $\tau$ ,
2.  $\tau_{i+1}$  is a  $\partial \vartheta_{i+1}$ -maximal extension of  $(\tau_i)_{\uparrow \partial \vartheta_i}$ , for  $i < n$ , and
3.  $\tau_n = (\tau_{n-1})_{\uparrow \partial \vartheta_{n-1}}$ .

$\vec{\vartheta}$ -polytopologies are weak versions of BG-polytopologies. For example, suppose  $\mathfrak{X}$  is a  $\{\omega_1\}$ -polytopology over the interval topology. Then  $\tau_1$  is a rank-preserving extension of  $\mathcal{I}_{\omega_1}$  obtained just by adding sets that would be already included in the corresponding rank-preserving extension of  $\mathcal{I}_{\omega_1}$  in any corresponding BG-space of length  $\geq \omega_1$  over  $\mathcal{I}_1$ . As above, we identify  $\vec{\vartheta}$ -polytopologies with ambiances whose algebras are the whole powerset.

**Theorem 5.3.** Let  $\vec{\vartheta}$  be an increasing sequence of nonzero ordinals. Denote by  $\text{GLP} \upharpoonright \vec{\vartheta}$  the fragment of GLP whose only modalities appear in  $\vec{\vartheta}$ .

1. (Soundness) All theorems of  $\text{GLP} \upharpoonright \vec{\vartheta}$  hold in every  $\vec{\vartheta}$ -space.
2. (Completeness) Let  $\Lambda \geq \vartheta_n$  be a limit ordinal and  $(X, \tau)$  be any scattered space of height  $\geq e^\Lambda 1$ . Suppose  $\varphi$  is consistent with  $\text{GLP} \upharpoonright \vec{\vartheta}$ . Then  $\varphi$  is satisfied on a  $\vec{\vartheta}$ -polytopology over  $(X, \tau_{\uparrow 1})$ .

The result also holds also for successor ordinals by replacing  $e^\Lambda 1$  with  $e^\Lambda \omega$ . In fact, this general version is what we will prove; the smaller bound in the statement of the theorem follows from the fact that

$$\lim_{\lambda \rightarrow \Lambda} e^\lambda \omega = \lim_{\lambda \rightarrow \Lambda} e^{\lambda+1} 1 = \lim_{\lambda \rightarrow \Lambda} e^\lambda 1 \leq \lim_{\lambda \rightarrow \Lambda} e^\Lambda 1 = e^\Lambda 1$$

These bounds are sharp (this follows from Lemma 4.3).

Notice that in Theorem 5.3.2, we satisfy the consistent formula on a  $\vec{\vartheta}$ -polytopology over  $(X, \tau_{\uparrow 1})$ . We cannot in general replace this with  $(X, \tau)$ —consider an ordinal with the initial-segment topology. It is still useful to consider

polytopologies of this sort. We will call  $\vec{\vartheta}$ -polytopologies over the initial-segment topology *improper*.

In the remainder of this section, we outline the proof of Theorem 5.3. Since it is only a minor modification of that of Theorem 4.20, we will mention the points that differ and omit some of the proofs if they carry on verbatim. For some details, we refer the reader to [9]. Soundness follows from the analog of Lemma 3.4, Lemma 5.5 below, which is proved using the analog of Lemma 3.3:

**Lemma 5.4.**  *$(X, \tau)$  is an  $\vartheta$ -maximal space if, and only if, for all  $x \in X$  whose rank  $\rho$  is such that  $\ell^\vartheta \rho > 0$  and all  $V \in \tau \cap [0, \rho)_0^\vartheta$ , one of the following:*

1.  $V \cup \{x\} \in \tau$ , or
2.  $\rho_\tau(U \cap V) < \rho$  for some  $\tau$ -neighborhood  $U$  of  $x$ .

**Lemma 5.5.** *Suppose  $(X, \tau)$  is  $\vartheta$ -maximal and  $\lambda \geq \vartheta$ . Then  $\{dA: A \subset X\} \subset \tau_{\uparrow \lambda}$ .*

The proofs of Lemmata 5.4 and 5.5 are the same as for limit-maximal spaces; they can also be found in [2]. It follows from Lemma 5.5 that all  $\vec{\vartheta}$ -polytopologies are  $\text{GLP}_n$ -ambiances. The proof for completeness follows the same strategy as before. We start with the pullback lemma.

**Lemma 5.6** (pullback). *Suppose  $\mathfrak{Y} = (Y, \mathcal{S}_0, \dots, \mathcal{S}_n)$  is a (possibly improper)  $\vec{\vartheta}$ -polytopology over  $(Y, \sigma)$  and  $f: (X, \tau) \rightarrow (Y, \sigma)$  is a  $d$ -map. Then, there exists a  $\vec{\vartheta}$ -polytopology  $\mathfrak{X} = (X, \mathcal{T}_0, \dots, \mathcal{T}_n)$  over  $(X, \tau)$  such that*

$$f: (X, \mathcal{T}_i) \rightarrow (Y, \mathcal{S}_i) \quad (5.2)$$

*is a  $d$ -map for each  $i \leq n$ .*

*Proof.* This is essentially the same proof as for the case  $\vartheta = 1$  (see [9, Lemma 8.5]). The following claim suffices:

**Claim 5.7.** *Suppose  $\mathcal{T}$  and  $\mathcal{S}$  are topologies such that  $f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$  is a  $d$ -map and  $\mathcal{S}'$  is a  $\vartheta$ -maximal extension of  $\mathcal{S}$ , then*

$$f^{-1}\mathcal{S}' := \{f^{-1}S: S \in \mathcal{S}'\}$$

*is a  $\vartheta$ -extension of  $\mathcal{T}$ . Moreover,  $f: (X, \mathcal{T}') \rightarrow (Y, \mathcal{S}')$  is a  $d$ -map, for any  $\vartheta$ -extension  $\mathcal{T}'$  of  $f^{-1}\mathcal{S}'$ .*

To see this suffices, suppose the claim holds. Then, letting  $\mathcal{T}_0$  be any  $\vartheta_0$ -maximal extension of  $f^{-1}\mathcal{S}_0$ , we obtain that (5.2) holds for  $i = 0$ . Inductively, suppose (5.2) holds for some  $i$ . By Lemma 4.2.3,

$$f: (X, \mathcal{T}_{i \uparrow \partial \vartheta_i}) \rightarrow (Y, \mathcal{S}_{i \uparrow \partial \vartheta_i})$$

is a  $d$ -map. By definition,  $\mathcal{S}_{i+1}$  is a  $\partial \vartheta_{i+1}$ -maximal extension of  $\mathcal{S}_{i \uparrow \partial \vartheta_i}$ , whereby the claim yields that (5.2) holds for  $i+1$  if we set  $\mathcal{T}_{i+1}$  to be some  $\partial \vartheta_{i+1}$ -maximal extension of  $f^{-1}\mathcal{S}_i$ . Hence, it suffices to prove the claim.

*Proof of the claim.* Let  $\mathcal{R} = f^{-1}\mathcal{S}'$ . Clearly,  $f: (X, \mathcal{R}) \rightarrow (Y, \mathcal{S}')$  is a d-map. It follows from the fact that

$$f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{S}) \text{ is a d-map} \quad (5.3)$$

that  $\mathcal{T} \subset \mathcal{R}$ , whence  $\mathcal{R}$  is a rank-preserving extension of  $\mathcal{T}$ . Let  $x \in X$  be such that  $\ell^\vartheta \rho_{\mathcal{T}} x = 0$ . We need to show that  $id: (X, \mathcal{T}) \rightarrow (X, \mathcal{R})$  is continuous at  $x$ . This follows from the fact that  $\mathcal{S}'$  is a  $\vartheta$ -extension of  $\mathcal{S}$ : for any  $\mathcal{R}$ -neighborhood of  $x$  of the form  $f^{-1}V$ , we have that  $V$  is a  $\mathcal{S}'$ -neighborhood of  $f(x)$  and  $\ell^\vartheta \rho_{\mathcal{S}'} f(x) = 0$ , so that  $f(x) \in U$  for some  $U \in \mathcal{S}$  with  $U \subset V$ . Therefore,  $x \in f^{-1}U \subset f^{-1}V$  and  $f^{-1}U \in \mathcal{T}$ , by (5.3).

Now let  $\mathcal{T}'$  be any  $\vartheta$ -extension of  $\mathcal{R}$ . Clearly,  $f: (X, \mathcal{T}') \rightarrow (Y, \mathcal{S}')$  is continuous and pointwise discrete. Suppose towards a contradiction that  $x \in X$  and  $W \in \mathcal{T}'$  witness a failure of  $f$  being open. Let

$$\rho := \rho_{\mathcal{T}} x = \rho_{\mathcal{T}'} x = \rho_{\mathcal{S}'} f(x).$$

Note that we must have  $0 < \ell^\vartheta \rho$ . Without loss of generality, we may assume  $\rho$  is the least possible rank of a counterexample and  $W$  contains no other point of rank  $\geq \rho$ , so that  $W = W_0 \cup \{x\}$ , for some  $W_0 \in \mathcal{T} \cap [0, \rho)_0^{\mathcal{T}}$ . We will arrive at a contradiction using Lemma 5.4: since  $f$  is rank-preserving, we have that  $\ell^\vartheta \rho > 0$  and  $f(W_0) \in \mathcal{S}' \cap [0, \rho)_0^{\mathcal{S}'}$ . Hence, by Lemma 5.4, one of the following:

1.  $f(W_0) \cup \{f(x)\} \in \mathcal{S}'$ , or
2.  $\rho_{\mathcal{S}'}(U \cap f(W_0)) < \rho$  for some  $\mathcal{S}'$ -neighborhood  $U$  of  $f(x)$ .

The latter must hold, since  $f(W_0) \cup \{f(x)\} = f(W)$  is not  $\mathcal{S}'$ -open by hypothesis. The key point is that

$$f^{-1}U \cap W = (f^{-1}U \cap W_0) \cup \{x\} = f^{-1}(U \cap f(W_0)) \cup \{x\}$$

is a  $\mathcal{T}'$ -neighborhood of  $x$  and so  $\rho_{\mathcal{T}'}(f^{-1}U \cap W) \geq \rho$ . However,

$$\rho_{\mathcal{T}'}(f^{-1}(U \cap f(W_0)) \cup \{x\}) = \rho_{\mathcal{S}'}(U \cap f(W_0)) \cup \{\rho\},$$

which is impossible by 2 above, because  $\rho$  is a limit ordinal.  $\square$

**Corollary 5.8.** *Suppose  $(X, \tau)$  is a scattered space and*

$$\mathfrak{Y} = ([1, \rho_\tau(X)], \mathcal{S}_0, \dots, \mathcal{S}_n)$$

*is a  $\vec{\vartheta}$ -polytopology over  $([1, \rho_\sigma(X)], \mathcal{I}_\varsigma)$ . Then, there exists a  $\vec{\vartheta}$ -polytopology  $\mathfrak{X} = (X, \mathcal{T}_0, \dots, \mathcal{T}_n)$  over  $(X, \tau_{\uparrow \varsigma})$  such that*

$$\ell^\varsigma \circ \rho_\tau: (X, \mathcal{T}_i) \rightarrow ([1, \rho_\tau(X)], \mathcal{S}_i)$$

*is a d-map for each  $i \leq n$ .*

*Proof.* Immediate from the preceding lemma and Lemma 4.13.  $\square$

Corollary 5.8 is still true in the degenerate case  $\varsigma = 0$ . In this case, notice that  $\mathcal{I}_0$  is already 1-maximal, for there is only one point of each rank. Using Lemma 5.6, we carbon-copy the proof of Lemma 4.21 (replacing 1-maximality with  $\vartheta_i$ -maximality and using the new pullback lemma) to obtain:

**Lemma 5.9.** Assume  $\varsigma$  is an additively indecomposable ordinal,  $([1, \kappa], \mathcal{T}_0, \dots, \mathcal{T}_n)$  and  $([1, \lambda], \mathcal{S}_0, \dots, \mathcal{S}_n)$  are  $\vec{\vartheta}$ -polytopologies over  $\mathcal{I}_\varsigma$ , and  $\{\kappa_0, \dots, \kappa_m\}$  is a finite subset of  $[1, \kappa]$ . Then, there exist:

1. An ordinal  $\Theta < e^{\varsigma + \vartheta_n} \omega$  and a partition of  $\Theta$  into two  $\mathcal{I}_{\varsigma+1}$ -clopen sets  $X_\uparrow$  and  $X_\downarrow$ .
2. A  $\vec{\vartheta}$ -polytopology  $([1, \Theta], \mathcal{R}_0, \dots, \mathcal{R}_n)$  over  $\mathcal{I}_\varsigma$ .
3. Functions  $\pi_0: [1, \Theta] \rightarrow [1, \kappa]$  and  $\pi_1: [1, \Theta] \rightarrow [1, \lambda]$  such that:
  - $\pi_0: (X_\downarrow, \mathcal{R}_i) \rightarrow ([1, \kappa], \mathcal{T}_i)$  is a d-map for each  $i$ ;
  - $\pi_1: (X_\uparrow, \mathcal{R}_i) \rightarrow ([1, \lambda], \mathcal{S}_i)$  is a d-map for each  $i > 0$ ;
  - $X_\uparrow \subset d\pi_0^{-1}(\kappa_i)$  for any  $i < m$ .

We use Lemma 5.9 to prove the following embedding lemma from which completeness follows (via Lemma 4.9, using Lemma 4.8 and Corollary 5.8 applied to  $\varsigma = 1$  to reduce to the case that  $X$  is an ordinal over some  $\mathcal{I}_\lambda$ ):

**Lemma 5.10.** Let  $(T, <_0, <_1, \dots, <_n)$  be a finite  $J_n$ -tree with root  $r$  and  $\vec{\vartheta}$  be an increasing  $n$ -sequence of nonzero ordinals. Then, for any  $\varsigma > 0$ , there exist

- a  $\vec{\vartheta}$ -polytopology  $([1, \Theta], \mathcal{T}_0, \dots, \mathcal{T}_n)$  over  $([1, \Theta], \mathcal{I}_\varsigma)$  such that  $\Theta < e^{\varsigma + \vartheta_n} \omega$ ; and
- a surjective  $J_n$ -map  $f: ([1, \Theta], \mathcal{I}_\varsigma) \rightarrow T$  such that  $f^{-1}(r) = \{\Theta\}$ .

*Proof.* The proof is by induction on  $n$ . The base case again follows from [3, Theorem 6.11], so we assume that the result holds for all  $m < n$  and proceed by a subsidiary double induction on

1.  $\varsigma$ , which we decompose as  $\varsigma_0 + \omega^\rho$ , and
2.  $\text{hgt}_0(T)$ , the height of  $<_0$ ,

in that order. The strategy is to proceed as in Lemma 4.18, except that we need to consider an additional case, since  $\varsigma$  might not be additively indecomposable. Let  $\text{hgt}_i(T)$  be the height of  $<_i$ .

**Case I:**  $\omega^\rho < \varsigma$ . By the induction hypothesis (over  $\varsigma$ ) applied to  $\omega^\rho$ , there are:

- a  $\vec{\vartheta}$ -polytopology  $([1, \Theta_0], \mathcal{S}_0, \dots, \mathcal{S}_n)$  over  $([1, \Theta_0], \mathcal{I}_{\omega^\rho})$  such that  $\Theta_0 < e^{\omega^\rho + \vartheta_n} \omega$ ; and
- a surjective  $J_n$ -map  $g: ([1, \Theta_0], \mathcal{I}_{\omega^\rho}) \rightarrow T$  such that  $g^{-1}(r) = \{\Theta_0\}$ .

By Lemma 4.13,

$$\ell^{\varsigma_0}: ([1, e^{\varsigma_0} \Theta_0], \mathcal{I}_\varsigma) \rightarrow ([1, \Theta_0], \mathcal{I}_{\omega^\rho})$$

is a d-map, whence by Lemma 5.6, there is a  $\vec{\vartheta}$ -polytopology

$$\mathfrak{X} = ([1, e^{\varsigma_0} \Theta_0], \mathcal{T}_0, \dots, \mathcal{T}_n)$$

over  $([1, e^{\varsigma_0} \Theta_0], \mathcal{I}_\varsigma)$  such that

$$\ell^{\varsigma_0}: ([1, e^{\varsigma_0} \Theta_0], \mathcal{T}_i) \rightarrow ([1, \Theta_0], \mathcal{S}_i)$$

is a d-map for each  $i \leq n$ . It follows that  $f := g \circ \ell^{\varsigma_0}$  is a d-map from  $\mathfrak{X}$  onto  $T$ . Moreover,

$$\Theta := e^{\varsigma_0} \Theta_0 < e^{\varsigma_0 + \omega^\rho + \vartheta_n} \omega = e^{\varsigma + \vartheta_n} \omega.$$

**Case II:**  $\varsigma$  is additively indecomposable and  $0 = \text{hgt}_0(T)$ , so that  $<_0 = \emptyset$ . Let  $\vec{\vartheta}^- = \vec{\vartheta} \upharpoonright [1, n]$  and  $\partial\vartheta_0 = -\varsigma + \vartheta_0$ . By induction hypothesis (on  $n$ ), there are:

- a  $\vec{\vartheta}^-$ -polytopology  $([1, \Theta_0], \mathcal{S}_1, \dots, \mathcal{S}_n)$  over  $([1, \Theta_0], \mathcal{I}_{\partial\vartheta_0})$  such that  $\Theta_0 <_{e^{\partial\vartheta_0 + \vartheta_n}\omega}$ ; and
- a surjective  $J_n$ -map  $g : [1, \Theta_0] \rightarrow T$  such that  $g^{-1}(r) = \{\Theta_0\}$ .

Notice that  $([1, \Theta_0], \mathcal{I}_0, \mathcal{S}_1, \dots, \mathcal{S}_n)$  is a(n improper)  $\vec{\vartheta}$ -polytopology and that

$$\ell^\varsigma : ([1, e^\varsigma\Theta_0], \mathcal{I}_\varsigma) \rightarrow ([1, \Theta_0], \mathcal{I}_0)$$

is a d-map. By Corollary 5.8, there exists a  $\vec{\vartheta}$ -polytopology  $\mathfrak{X}$  of the form  $([1, \Theta], \mathcal{T}_0, \dots, \mathcal{T}_n)$  over  $([1, \Theta], \mathcal{I}_\varsigma)$  such that

$$\ell^\varsigma : ([1, e^\varsigma\Theta_0], \mathcal{T}_i) \rightarrow ([1, \Theta_0], \mathcal{S}_i)$$

is a d-map for each  $i \leq n$ . Let  $\Theta := e^\varsigma\Theta_0 < e^{\varsigma + \vartheta_n}\omega$ . Let  $f := g \circ \ell^\varsigma$ . We have that

$$f : ([1, \Theta], \mathcal{T}_1, \dots, \mathcal{T}_n) \rightarrow (T, <_1, \dots, <_n) \quad (5.4)$$

We claim 5.4 holds for the full space  $\mathfrak{X}$  and the structure  $(T, <_0, \dots, <_n)$ , i.e.,  $f$  is already a  $J_n$ -map: condition  $(j_1)$  is given by definition;  $(j_2)$  is satisfied trivially since the topology induced by  $<_0$  is discrete;  $(j_3)$  and  $(j_4)$  hold because  $f$  is a d-map and  $r$  is the sole hereditary 1-root of  $T$ .

**Case III:**  $\varsigma$  is additively indecomposable and  $0 < \text{hgt}_0(T)$ . This is proved exactly as Case II of Lemma 4.18.

Since we have considered all cases, the lemma follows.  $\square$

## 6 Closing remarks

We have shown that the logic GLP, albeit relationally incomplete, is topologically complete. However, some related questions remain open. *Is GLP strongly complete with respect to any nice (topological or otherwise) semantics?* Here, by *strong completeness* (with respect to a class of models  $\mathcal{X}$ ), we mean the following assertion: whenever  $\Gamma$  is a set of  $\mathcal{L}_\Lambda$ -sentences consistent with  $\text{GLP}_\Lambda$ , then there is some model  $\mathfrak{X} \in \mathcal{X}$  where  $\Gamma$  is satisfied. This question is partially a motivation for the choice of Icard topologies as the skeleta of our models, given that GLP cannot be strongly complete for other natural topologies. We give the following example.

**Proposition 6.1.** *Suppose  $X$  is a scattered space such that every  $G_\delta$  set is open. Then GL is not strongly complete with respect to  $X$ .*

*Proof.* This is a generalization of the usual proof that GL is not strongly complete with respect to trees. Let

$$\Gamma = \{\Diamond p_0\} \cup \{\Box(p_i \rightarrow \Diamond p_{i+1}) : i < \omega\}.$$

Suppose  $\Gamma$  is satisfied at some  $x$ . Then, for each  $i$ , there is a punctured neighborhood  $U_i$  of  $x$  such that any point satisfying  $p_i$  is a limit of points satisfying

$p_{i+1}$ . Let  $U = \bigcap_{i < \omega} U_i$  be open and nonempty. Since  $x$  satisfies  $\Diamond p_0$ ,  $U$  contains some  $x_0$  of rank  $\alpha_0$  satisfying  $p_0$ . Inductively, for each  $i < \omega$ , there is some  $x_i \in U$  satisfying  $p_i$  and, by

$$\Box(p_i \rightarrow \Diamond p_{i+1}),$$

$U$  contains some  $x_{i+1}$  of rank  $\alpha_{i+1} < \alpha_i$  satisfying  $p_{i+1}$ . This gives an infinite decreasing sequence of ordinals.  $\square$

**Corollary 6.2.** *GL is not strongly complete with respect to topologies given by countably complete filters, such as the club topology.*

One question that was left unanswered was that of the topological completeness of  $\text{GLP}_{\text{Ord}}$ . We remark, however, that this would follow from the constructions in this paper if the d-maps (5.1) were proved to exist. This raises the question: *for an additively indecomposable  $\kappa$ , is there a d-map reducing the topology  $\tau^\kappa$  to  $\mathcal{I}_\kappa$ ?*

A final open problem remains: *is  $\text{GLP}_\Lambda$  complete with respect to the canonical  $\text{GLP}_\Lambda$ -spaces, under suitable set-theoretic assumptions?* Although the question is unanswered as of now, doing so positively seems plausible [4]. It is also a very reasonable question, given that these spaces are natural-looking and appealing in their own right, as evidenced e.g., by the work of J. Bagaria, M. Magidor, and H. Sakai [5] and H. Brickhill [13].

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